Fractal Tilings

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Abstract

In this paper we will discuss how to create fractal tilings from a few simple matrices and formulas. We will do this by drawing several examples of the tilings and discussing how we got there.

Introduction

In this paper we will be generating tilings with tiles called *fractiles* whose boundaries are fractal curves. Fractal curves are objects or quantities that display self-similarity, in a somewhat technical sense, on all scales. This means that it looks the same at any scale. We will use an iterative process, involving repeated compositions of two or more functions and those, in turn, will generate the fractal tiling.

Examples of Fractal Tilings

When we start with a matrix $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are chosen so that $a^2 + b^2 > 1$. We must understand that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ are points in the complex plane and $M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 - bx_2 \\ ax_1 + bx_2 \end{bmatrix}$ represents the complex multiplication of $x_1 + ix_2$ by $a + ib$. Next, we must find a collection of vectors that will translate the copies of the *fractile* so that they are positioned correctly in the tiling. The unit square that is determined by the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is mapped onto the square S with area $m = a^2 + b^2$ and is spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$. We will define the set $\xi = \{\mathbf{r}_j\}$ and the vectors in this set have integer coordinates that lie in or on S but not on the two outer edges that don't have the origin as a vertex. ξ has exactly m vectors. We will generate an example using the iterative process that will be explained in a later section.

Example 1

In this first example, let $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ the $m = 2$. Using Figure 1, we can determine that the two translation vectors are $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Now we

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have $\xi = \{\mathbf{r}_1, \mathbf{r}_2\}$. For $z = (x_1, x_2)$, where z is our initial point of translation, we can define our mappings as $f_j(\mathbf{z}) := \mathbf{r}_j + M^{-1}(\mathbf{z})$ for $j = 1, 2$ That is,

$$f_1 := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} .5 & -.5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$f_2 := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} .5 & -.5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The collections of functions $\{f_j\}$ is called an *iterated function system*. To initiated this process a initial point z_o is randomly selected in the plane and is used to evaluate $f_1(z_o)$ and $f_2(z_o)$. For $n \geq 1$, we make sure to choose recursively and randomly so that $z_n \epsilon \{f_1(z_{n-1}), f_2(z_{n-1})\}$. Points will be lying near the tiling after a few iterations, but thousands of iterations will be needed to generate the desired tiling. The set A of randomly selected points that result is called the *attractor*. $A = A_1 \cup A_2$, where $A_j = f_j A$. The result of the iterated function system for this example can be seen in Figure 2.

Example 2

If we have $M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. These vectors, seen in Figure 3, will produce three tiles that are stacked horizontally in Figure 4.

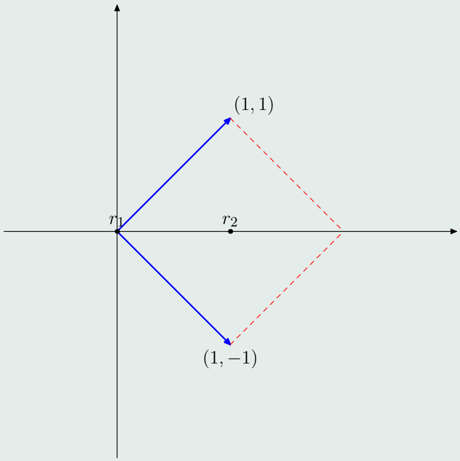

Figure 1: Finding Equivalent Residue Vectors.

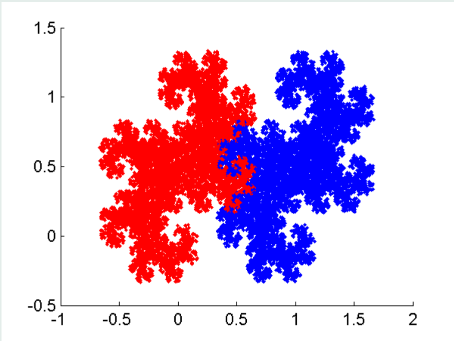

Figure 2: Snowflake Spiral.

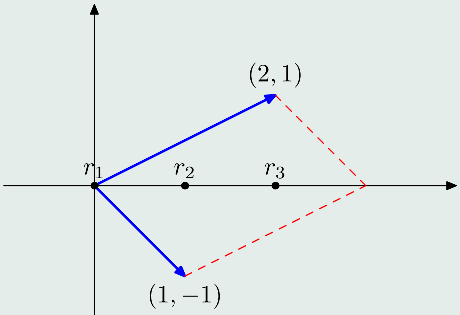

Figure 3: Finding Equivalent Residue Vectors.

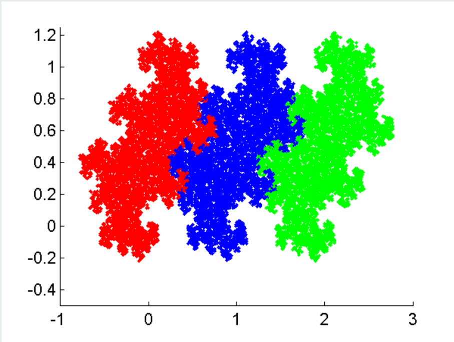

Figure 4: Horizontal Tiling.

Creating the Tilings

To generate a tiling we need a matrix to be an invertible integer matrix that is an *expansive map*, i.e. all eigenvalues have modulus larger than 1. The matrix we will choose will be $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The translation vectors are chosen with the following process. For a matrix M as above, $|det(M)| = |ad - bc| = m$ is the area of parallelogram P spanned by the two vectors $\mathbf{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$. These vectors are called *principal residue vectors*. The vectors in $\{\mathbf{r}_j\}$ form a *complete residue system* for M.

Let L be the lattice of points in the complex plane with integer coordinates (these points are known as Gaussian Integers). For $j = 1, ..., m$, define $L_j := \{\mathbf{r}_j + Mx : x \epsilon L\}$. We see now, that the corner points of the parallelogram P are members of L and are linear combinations of each of the columns of our matrix M. These integer linear combinations form a subset G of L, which in turn forms a grid of parallelograms, and each are congruent to P. Each point $\mathbf{r}_j$ of L inside P is equivalent to one point $\mathbf{y}_j (\mathbf{r}_j \approx \mathbf{y}_j)$ inside each of these congruent parallelograms.

Generally, as long as $\mathbf{y}_1 = \mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{y}_j \approx \mathbf{r}_j$ for $j = 2, ...m$, then the collection of vectors $\{\mathbf{y}_j\}$ will also form a complete residue system for matrix M. The location of the residue vectors determines the locations of the fractiles but the shape of the tilings may change drastically with the different choices of residue systems.

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Short Summary of Important Ideas

M represents an *expansive map*, $\{\mathbf{y}_1, ...\mathbf{y}_m\}$ is a *complete residue system* for M, and $f_j(\mathbf{z}) := \mathbf{r}_j + M^{-1}(\mathbf{z})$. The attractor set $A = \cup_m^{j=1} A_j$ is the tiling of m tiles A_j. These tiles are now called *m-rep tiles*. These ideas will now be used to create a tiling of m-rep tiles.

Example 3

Let $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$; then $m = 5$. Here the principal residue vectors are

$$r_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad r_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For a more symmetric tiling, we choose the following equivalent residue vectors for our residue system out of the collection $\{\mathbf{y}_j\}$. The Figure 6 is created by using

$$\mathbf{y}_1 = \mathbf{r}_1, \quad \mathbf{y}_2 = \mathbf{r}_2, \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \approx \mathbf{r}_3, \quad \mathbf{y}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \mathbf{r}_4, \quad \text{and} \quad \mathbf{y}_5 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \approx \mathbf{r}_5.$$

The vectors $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5\}$ are symmetric about $\mathbf{r}_1$.

Tiles with Radial Symmetry

When $m = 2, 3, 4, 5,$ and 7, we are able to create a tiling that has radial symmetry.

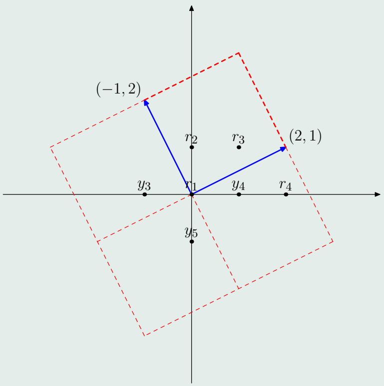

Figure 5: Finding Equivalent Residue Vectors.

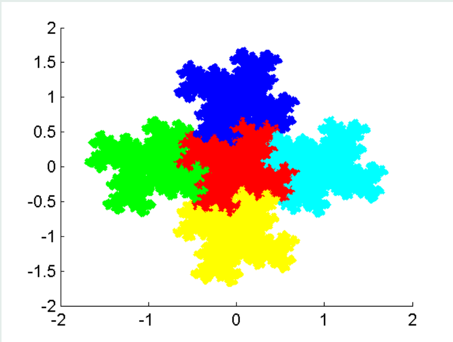

Figure 6: 5-rep tile.

Example 4

Let our matrix $M = \begin{bmatrix} 2 & -2 \\ 2 & 0 \end{bmatrix}$. To get radial symmetry in our tiling we need to generate a tiling using three vectors that are symmetrically located around $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Our complete residue system for M is $\mathbf{y}_1 = \mathbf{r}_1, \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \mathbf{r}_2, \mathbf{y}_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \approx \mathbf{r}_3$, and $\mathbf{y}_4 = \mathbf{r}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This residue system is shown in Figure 7 Even though they form a complete residue system, these vectors are not symmetric about $\mathbf{r}_1$. However, we see that the complex third roots of unity are $\mathbf{v}_1 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and$\mathbf{v}_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$ but these vectors are not Gaussian integers. If we then use the linear transformation of $B = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$ and we get that $B\mathbf{y}_1 = \mathbf{y}_1, B\mathbf{y}_2 = \mathbf{v}_2, B\mathbf{y}_3 = \mathbf{v}_3$, and$B\mathbf{y}_4 = \mathbf{v}_1$. This new iteration process is using the new function

$$f_j(\mathbf{z}) = B\mathbf{y}_j + h^{-1}(\mathbf{z})$$

where $h = BMB^{-1}$ and $h^{-1} = \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix}$. This transformation B is the change of basis matrix that converts the lattice formed by ξ into one formed by $\mathbf{v}_1$and$\mathbf{v}_2$.

Here seen in Figure 8.

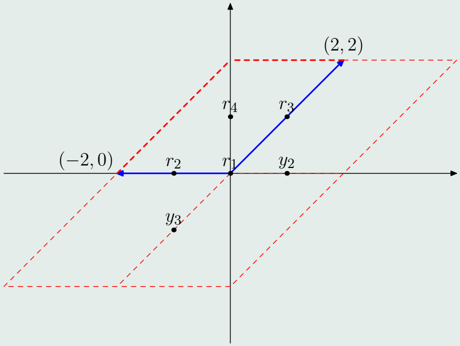

Figure 7: Finding Equivalent Residue Vectors.

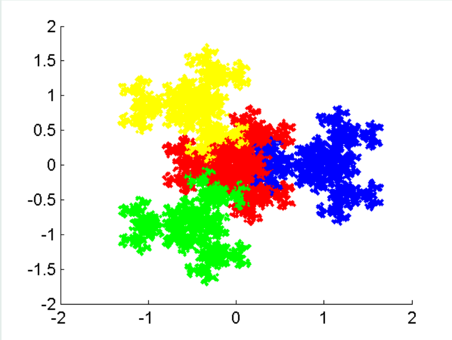

Figure 8: 4-rep Tile with Symmetry.

Example 5

Here is an example of six fractiles symmetrically located about the first and center tile. $M = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ and we have the vectors $\mathbf{r}_j$ as shown in Figure 9 but we can choose residue vectors $\mathbf{y}_1 = \mathbf{r}_1, \mathbf{y}_2 = r_2, \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \approx \mathbf{r}_3, \mathbf{y}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \approx \mathbf{r}_4, \mathbf{y}_5 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \approx \mathbf{r}_5, \mathbf{y}_6 = \mathbf{r}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and$\mathbf{y}_7 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \mathbf{r}_7$. We set $B = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$ and we iterate with the function $f_j(\mathbf{z}) = B\mathbf{y}_j + h^{-1}(\mathbf{z})$ for

$j = 1, 2, ..., 7$ where $h^{-1} = B^{-1}M^{-1}B = \begin{bmatrix} 2/7 & \sqrt{3}/7 \\ -\sqrt{3}/7 & 2/7 \end{bmatrix}$. The resulting tiling can be seen in Figure 10.

Similarity Maps

A similarity map g will satisfies $|g(\mathbf{x}) - g(\mathbf{y})| = r|\mathbf{x} - \mathbf{y}|$, for $r > 0$ and all $\mathbf{x}$,$\mathbf{y}$ in the plane. Similarity maps, geometrically are a composition of any collection of four simple mappings:

- Scaling by a positive factor r
- Rotation about the origin
- Translation, and
- Reflection

Our attractor A is self-similar if each mapping in the collection $\{f_j\}$ is a similarity map with $0 < r < 1$. To be self-similar, A is the union of m smaller copies of itself. The attractor has the same shape as the individual m-rep tiles. In examples 2,4,5, and 6, these rep-tiles are formed by the linear map: multiplication by a complex number.

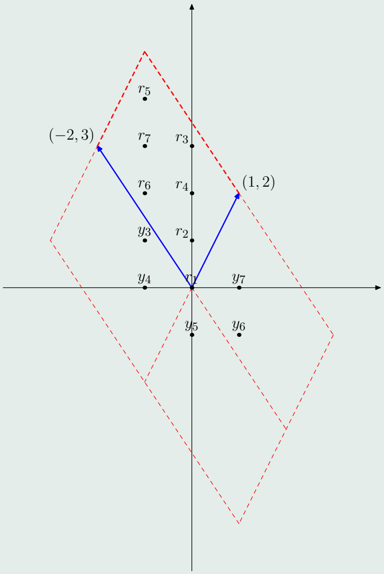

Figure 9: Finding Equivalent Residue Vectors.

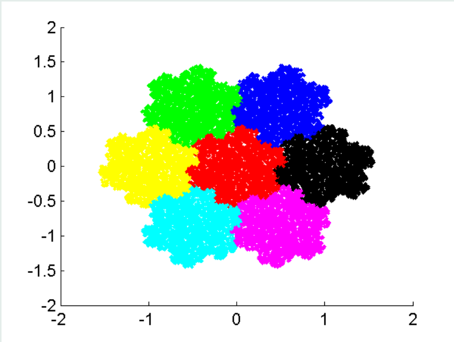

Figure 10: Gosper Snowflake.

Developing similarity maps

In our Examples 4 and 5, there was a change of basis that was applied to M and that produced our similarity mappings and attractive tilings. This works if M satisfies either of two conditions

- M has two real eigenvalues with independent eigenvectors, or
- M has a pair of complex conjugate eigenvalues.

If M satisfies the first condition, let λ_1 and $\lambda_2 (\lambda_1 = \pm\lambda_2)$ be real eigenvalues for M with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. We will let $B^{-1} = [\mathbf{v}_1, \mathbf{v}_2]$ be the matrix with column vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Now then,

$$MB^{-1} = M[\mathbf{v}_1, \mathbf{v}_2] = [\lambda_1 \mathbf{v}_1, \lambda_2 \mathbf{v}_2] = B^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

so that $h := BMB^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is a similarity map. This is needed because then h is then used to create the tilings with the function $f_j = B\mathbf{y}_j + h^{-1}(\mathbf{z})$, where $\{\mathbf{y}_j\}$ are residue vectors for M.

With this information, we are able to determine that h only depends on the eigen*values*, while the translation vectors, and the tiling, vary with the eigen*vectors*.

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Example 6

In this example, the matrix $M = \begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix}$ has a determinant -6 and eigenvalues $\lambda = \pm\sqrt{6}$. The eigenvectors of M are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ (\sqrt{6}-2)/2 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -(\sqrt{6}+2)/2 \end{bmatrix}$ which means that $B^{-1} = \begin{bmatrix} 1 & 1 \\ (\sqrt{6}-2)/2 & -(\sqrt{6}+2)/2 \end{bmatrix}$. Then the similarity map is $h = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & -\sqrt{6} \end{bmatrix}$ The tiling generated using the functions $f_j(\mathbf{z}) = Br_j + h^{-1}(\mathbf{z})$ can been seen in Figure 11. The principal residue vectors are

$$\mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{r}_4 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{r}_5 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{r}_6 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Example 7

Start with matrix $M = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$. This M has a determinant 3 and eigenvalues $3/2 \pm i\sqrt{3}/2$. Because we have our $B = \begin{bmatrix} 1 & 1/2 \\ 0 & -\sqrt{2}/2 \end{bmatrix}$, the our similarity mapping $h = BMB^{-1}$ is given by the matrix $\begin{bmatrix} 3/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{bmatrix}$. Next, choose the residue

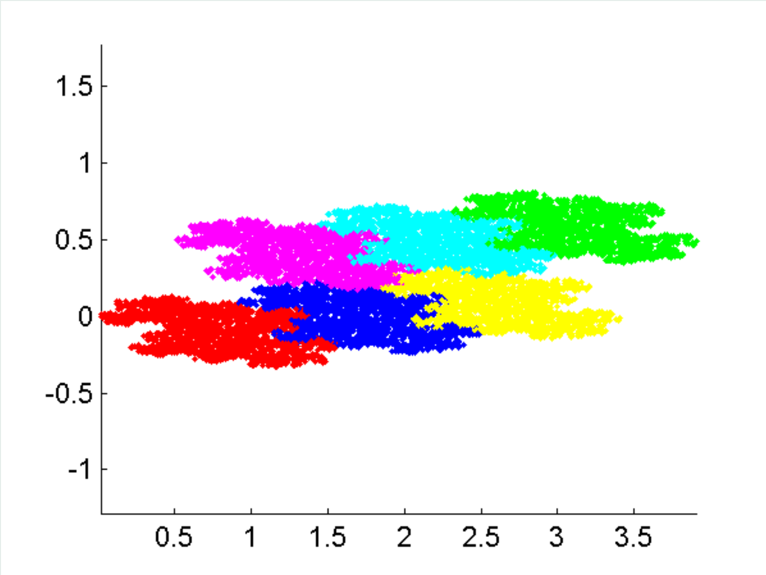

Figure 11: Similarity Tiling using a change of basis.

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vectors $\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and iterate with the functions $f_j = B\mathbf{y}_j + h^{-1}(\mathbf{z})$, and $j = 1, 2, 3$. The resulting tiling is a *terdragon* and is a 3-rep tile and is seen in Figure 12.

Variations

When working with these fractal tilings, one is able to vary the look of the tiling by merely omitting a function with various residue vectors or changing the main matrix M but keeping the residue vectors. These small changes will create similar but different tilings.

Example 8

We are able to generate the tiling in Figure 14 by using the function from Example 5 (Figure 10) and simply removing the function that coincides with the residue vector $\mathbf{r}_1$.

Conclusion

In this paper, we've discussed the different steps that needed to be taken in order to create different and beautiful fractal tilings in the plane. We've given many examples of these tilings and have shown the different residue vectors and how to find them. This was an interesting paper to write and we hope that it was also interesting to read.

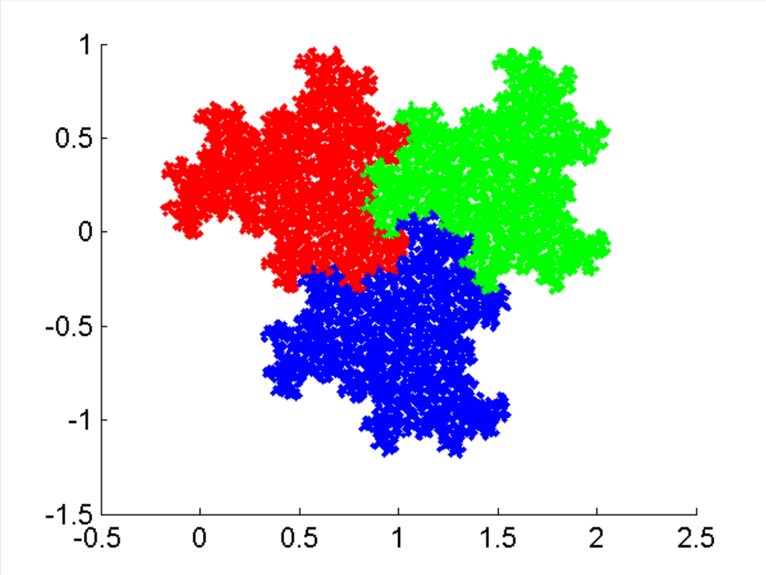

Figure 12: The Tergragon(a 3-rep tile).

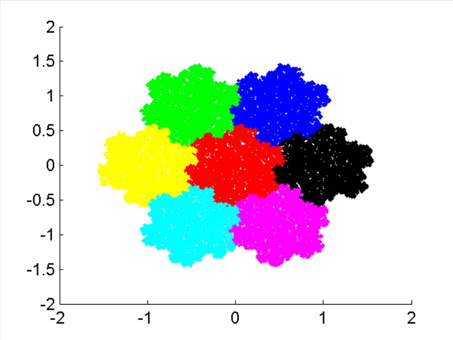

Figure 13: Gosper Snowflake.

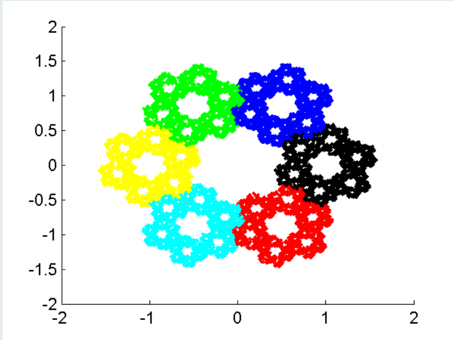

Figure 14: Modified Snowflake.

Acknowledgements

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