

The Catenary

Daniel Gent

December 10, 2008

History of the catenary.

- ▶ The catenary is defined by the shape formed by a hanging chain under normal conditions. The name catenary comes from the latin word for chain *catena*.

History of the catenary.

- ▶ The catenary is defined by the shape formed by a hanging chain under normal conditions. The name catenary comes from the latin word for chain *catena*.
- ▶ The shape of the catenary was originally thought to be a parabola by Galileo, but this was proved false by the German mathematician Jungius in 1663. It was not until 1691 that the true shape of the catenary was discovered to be the hyperbolic cosine in a joint work by Leibnez, Huygens, and Bernoulli.

Derivation of the catenary.

- ▶ The approach used by Leibnez, and his coworkers was as follows.

Derivation of the catenary.

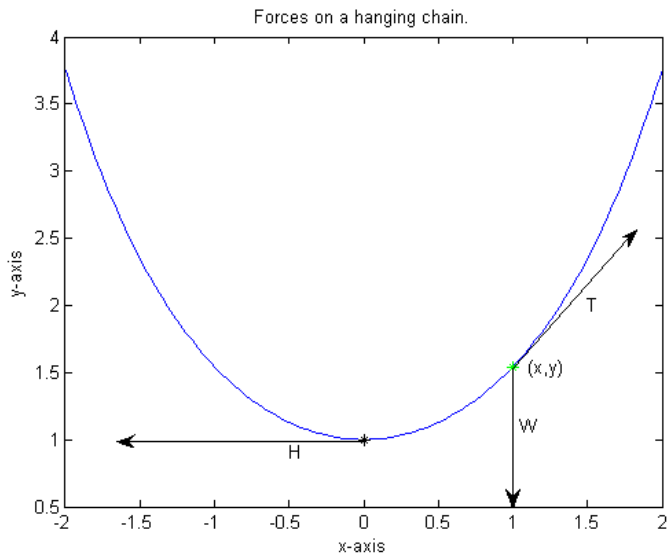
- ▶ The approach used by Leibnez, and his coworkers was as follows.
- ▶ The forces on the chain at the point (x,y) will be the tension in the chain (T) which is tangent to the chain, the weight of the chain (W), and the horizontal pull (H) at the origin. Now the magnitude of the weight is proportional to the distance (s) from the origin to the point (x,y) .

Derivation of the catenary.

- ▶ The approach used by Leibnez, and his coworkers was as follows.
- ▶ The forces on the chain at the point (x,y) will be the tension in the chain (T) which is tangent to the chain, the weight of the chain (W), and the horizontal pull (H) at the origin. Now the magnitude of the weight is proportional to the distance (s) from the origin to the point (x,y) .
- ▶ Thus the sum of the forces W and H must be equal to $-T$, but since T is tangent to the chain the slope of $H + W$ must equal $f'(x)$. Therefore

$$f'(x) = \frac{\|W\|}{\|H\|} = \frac{\mu s}{\|W\|} \text{ (where } \mu \text{ is the weight density)}$$

The hanging chain.



Derivation of the catenary.

- ▶ Now this may be written.

$$f'(x) = \frac{\mu}{\|W\|} \int_0^x \sqrt{1 + (f'(t))^2} \text{(by the formula for arc length.)}$$

Derivation of the catenary.

- ▶ Now this may be written.

$$f'(x) = \frac{\mu}{\|W\|} \int_0^x \sqrt{1 + (f'(t))^2} \text{ (by the formula for arc length.)}$$

- ▶ This is the same as the second order differential equation

$$f''(x) = \alpha \sqrt{1 + (f'(t))^2} \text{ (where } \alpha = \frac{\mu}{\|W\|} \text{.)}$$

Derivation of the catenary.

- ▶ Now this may be written.

$$f'(x) = \frac{\mu}{\|W\|} \int_0^x \sqrt{1 + (f'(t))^2} \text{(by the formula for arc length.)}$$

- ▶ This is the same as the second order differential equation

$$f''(x) = \alpha \sqrt{1 + (f'(t))^2} \text{(where } \alpha = \frac{\mu}{\|W\|} \text{.)}$$

- ▶ This solves to

$$f(x) = -\frac{1}{\alpha} + \frac{1}{\alpha} \cosh(\alpha x).$$

Derivation of the catenary.

- ▶ Now, although it is difficult to arrive at this solution check to ensure that it holds is an easy matter.

Derivation of the catenary.

- ▶ Now, although it is difficult to arrive at this solution check to ensure that it holds is an easy matter.
- ▶ First take the first and second derivatives.

$$f'(x) = \sinh(\alpha x)$$

$$f''(x) = \alpha \cosh(\alpha x)$$

Derivation of the catenary.

- ▶ Now, although it is difficult to arrive at this solution check to ensure that it holds is an easy matter.
- ▶ First take the first and second derivatives.

$$f'(x) = \sinh(\alpha x)$$

$$f''(x) = \alpha \cosh(\alpha x)$$

- ▶ Now input these to our differential equation.

$$(\alpha \cosh(\alpha x)) = \alpha \sqrt{1 + (\sinh(\alpha x))^2}$$

$$\alpha \cosh(\alpha x) = \alpha \sqrt{\cosh^2(\alpha x)}$$

$$\alpha \cosh(\alpha x) = \alpha \cosh(\alpha x)$$

Derivation of the catenary.

- ▶ Now, although it is difficult to arrive at this solution check to ensure that it holds is an easy matter.
- ▶ First take the first and second derivatives.

$$f'(x) = \sinh(\alpha x)$$

$$f''(x) = \alpha \cosh(\alpha x)$$

- ▶ Now input these to our differential equation.

$$(\alpha \cosh(\alpha x)) = \alpha \sqrt{1 + (\sinh(\alpha x))^2}$$

$$\alpha \cosh(\alpha x) = \alpha \sqrt{\cosh^2(\alpha x)}$$

$$\alpha \cosh(\alpha x) = \alpha \cosh(\alpha x)$$

- ▶ Thus $f(x) = -\frac{1}{\alpha} + \frac{1}{\alpha} \cosh(\alpha x)$ is a solution.

Properties of the catenary.

- ▶ A square can roll smoothly without slippage on a series of inverted catenaries.

Properties of the catenary.

- ▶ A square can roll smoothly without slippage on a series of inverted catenaries.
- ▶ In fact any normal polygon can roll on a series of catenaries. Though a triangle will not work in a real life system even though it should in theory.

Properties of the catenary.

- ▶ A square can roll smoothly without slippage on a series of inverted catenaries.
- ▶ In fact any normal polygon can roll on a series of catenaries. Though a triangle will not work in a real life system even though it should in theory.
- ▶ A second way to derive the catenary is to trace the vertex of a parabola as it rolls along the x-axis.

The catenary with complex numbers.

- ▶ One of the more odd properties of the catenary is how it behaves when numbers with only an imaginary part are the inputs. Ex: $\cosh ix$

The catenary with complex numbers.

- ▶ One of the more odd properties of the catenary is how it behaves when numbers with only an imaginary part are the inputs. Ex: $\cosh ix$
- ▶ Recall that $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
Also recall Euler's equation $e^{i\theta} = \cos \theta + i \sin \theta$

The catenary with complex numbers.

- ▶ One of the more odd properties of the catenary is how it behaves when numbers with only an imaginary part are the inputs. Ex: $\cosh ix$
- ▶ Recall that $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
Also recall Euler's equation $e^{i\theta} = \cos \theta + i \sin \theta$
- ▶

$$\begin{aligned}\text{Thus } \cosh(i\theta) &= \frac{(\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)}{2} \\ &= \frac{2 \cos \theta}{2} \\ &= \cos \theta\end{aligned}$$

The catenary with complex numbers.

- ▶ One of the more odd properties of the catenary is how it behaves when numbers with only an imaginary part are the inputs. Ex: $\cosh ix$
- ▶ Recall that $\cosh(x) = \frac{e^x + e^{-x}}{2}$.
Also recall Euler's equation $e^{i\theta} = \cos \theta + i \sin \theta$



$$\begin{aligned}\text{Thus } \cosh(i\theta) &= \frac{(\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)}{2} \\ &= \frac{2 \cos \theta}{2} \\ &= \cos \theta\end{aligned}$$

- ▶ A similar process will show that $\sinh i\theta = i \sin \theta$.