The Spherical Spiral

Chris Wilson
and
Geoff Zelder

Multivariable Calculus-50C
College of the Redwoods

08 Dec. 2008

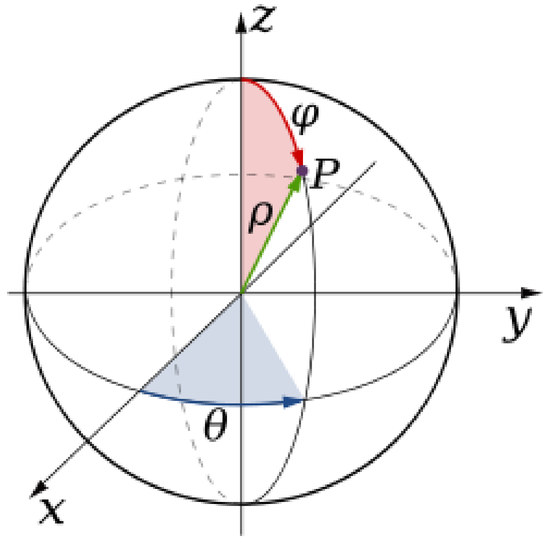
Abstract

Just about everyone is familiar with sirals in two dimensions but not often is a spiral seen as navigating a 3 dimensional sphere. Suprisingly there a several groundbreaking applications of the spherical spiral and with minor adjustments many three dimensional curves can be obtained from the basic equation of the spherical spiral.

Spherical Coordinates

The spherical spiral is very similar to a sphere in spherical coordinates and actually relatively simple using spherical coordinates. The basic equations for spherical coordinates are:

$$
\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\phi
\end{aligned}
$$



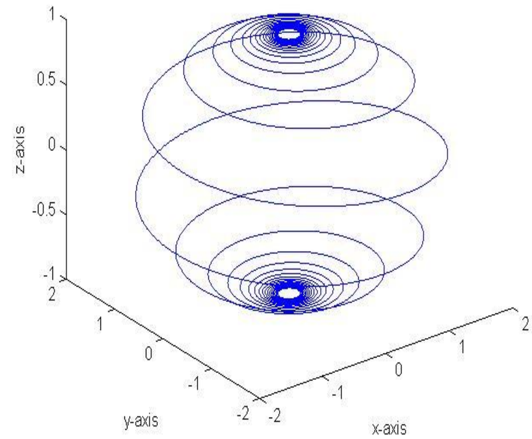
By letting ϕ run from 0 to π, and by letting θ run from 0 to 2π this creates a perfect shpere.

The Spherical Spiral

The spiral is very similar in it's equations as we let,

$$
\begin{aligned}
x &= \rho \cos\theta \cos\phi \\
y &= \rho \sin\theta \cos\phi \\
z &= -\sin\phi
\end{aligned}
$$

In this case we let θ run from 0 to $k\pi$ where the greater k is the more times the spiral will circumnavigate the sphere. We let $\phi = \tan^{-1} a\beta$, where a determines how far apart the spirals are and β determines how high up and down the spirals start. In a sense a determines and increment of angle that ϕ will range through and the β determines the start and stop angles of ϕ.



Parametrizing the Spherical Spiral

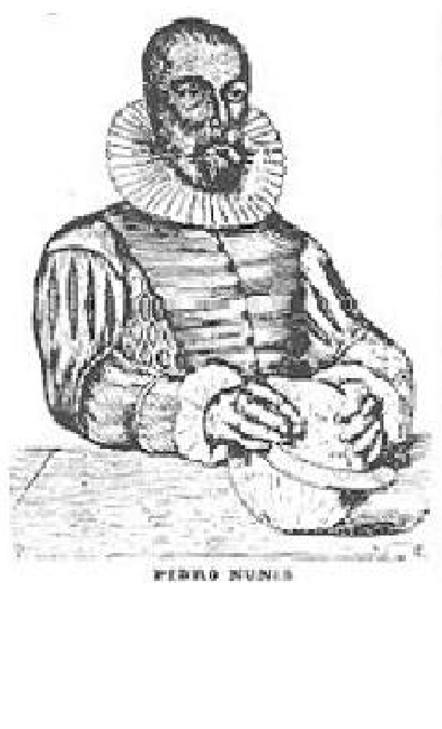
The spherical sprial can be parametretrized into terms of t and finally we arrive at,

$x = \frac{\cos t}{\sqrt{1+\alpha^{2t^2}}}$
$y = \frac{\sin t}{\sqrt{1+\alpha^{2t^2}}}$
$z = \frac{\alpha t}{\sqrt{1+\alpha^{2t^2}}}$

Here α is the angle along the longitude and t is the legth of the trip.

Some History and Applications

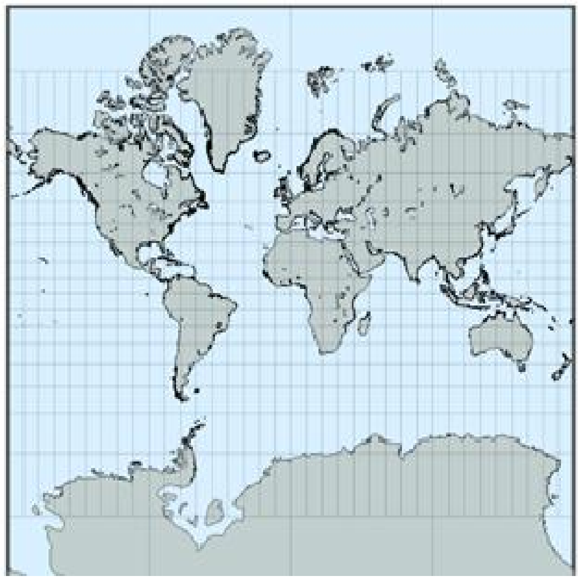
The spiral can be traced back to the time of Archimedes, but it was not until Papus considers curves of double curvature. Papus was able to create a spherical spiral by taking a point moving uniformly along the circumference of a great circle of a sphere, while the circumference moves uniformly around it's diameter. Eighteen hundred years later the first big application of the sherical spiral was found by Pedro Nunes, a sixteenth century cosmographer from Portugal.



PEDRO NUNES

His work with early navigators came up with two possible routes across the Atlantic. One route being the Great Circle Route where the shortest distance between two points on a sphere is not a straight line, but an arc that starts off heading in a northern direction and then comes back down. Nunes was asked by Martim Afonso de Sousa, founder of the Portuguese colonies in Brazil, to find the best route across the Atlantic, and in what direction should he travel. This is when Nunes came up with the two possible routes. The other route Nunes came up with was a straight line at an angle off the cardinal directions which is a loxodrome, or a spherical spiral.

Another application that came about from the work Nunes did with the loxodrome is the creation of the Mercator projection. Gerardus Mercator wanted to create maps of the globe on a flat surface and with the help of Nunes he was able to draw the map where the straight lines across the map were in fact loxodromes.

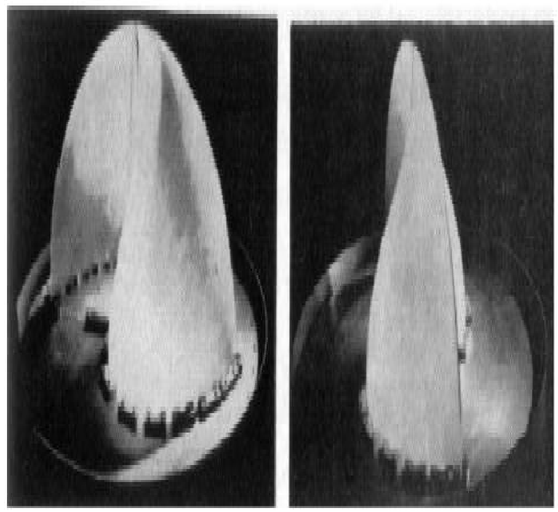


Navigators found this acceptable and easy to understand because of the recent paradox shift of the earth being round and not flat. The drawback was that the distance between the lines across the maps increased as they neared the poles causing the distortion of the continents to the far north and south. Navigators also accepted this because the maps were much easier to use. The Mercator projection is still being used today and even though maps are more accurate and detailed the basic concept from three-hundred years ago is still the same.

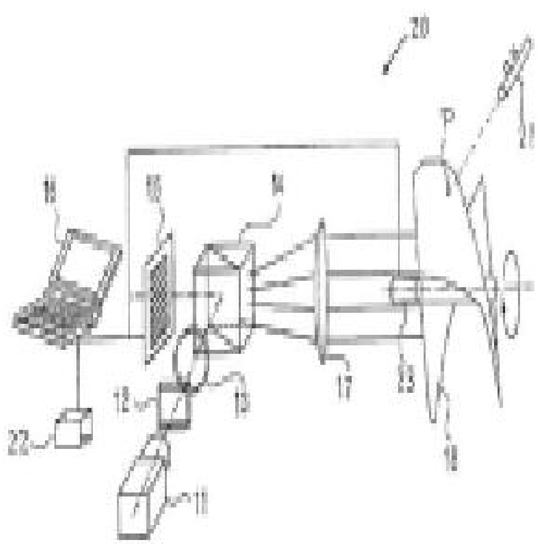
The first illustration of a spherical spiral was over three-hundred years after Nunes came up with this concept of the loxodrome. M.C. Esher, a twentieth century illustrator, unknowingly drew the Bolspiralen, a spherical spiral that best portrayed the geometrical meaning of Nunes' work with the loxodrome.

This drawing clearly shows that when traveling around a sphere on a straight line at an angle off the cardinal directions you will be taking the route of a spiral. This is exactly what Nunes was saying. If you are traveling on a globe that is completely covered with water and headed in a constant direction you would not come back to the same spot but travel in a spiral from pole to pole.

The following picture is of a 3D radar display which was proposed by D.W. Perkins in 1962, where he introduced a spherical spiral display with a specially shaped screen that rotates about a vertical axis and has a light beam that is projected up from below that produces a blip on the display much like a typical radar display does.



Like the sphere, the spherical spiral is the locus of circles with a common center and distributed about an axis that is the diameter of all the circles. In addition, each circle of the spherical spiral has a radius directly proportional to its angular position above the common axis. As the developed figure is rotated about its vertical axis at a constant velocity, the radius of the surface intersection with a plane containing the axis varies linearly with time. The high intensity beam is projected through an optical system and onto mirrors at the center. The mirrors direct the beam to any azimuth and elevation, also by using a shutter with three slits the display can produce three seperate targets along a common azimuth and elevation. Similar propositions have been made using computers to compose 3D sets of helical slices that generate a series of 2D images on a reflective surface of a light modulator. The series of 2D images are projected into a volumetric 3D-space display using a light source and projection optics. Voxels in 3-space are illuminated for each 2D projected image, each voxel being located at its corresponding spatial location. A pulse from a laser pointer creates a bright voxel within the display and is sychronized with the rotating helix. This creates a user controlled orientation in the 3D image. Optical encoders provide synchronization signals of the rotating helical display.



Bibliography

Cajori, Florian. "A History of Mathematics." New York: The Macmillan Company, 1919.

"Felix 3D Display" $\langle http://felix.cixforce.de/hof/hof.php5?show=curved\rangle$

"Mercator projection" $\langle http://en.wikipedia.org/wiki/Mercator_projection\rangle$

"Pedro Nunes" $\langle http://en.wikipedia.org/wiki/Pedro_Nunes\rangle$

"Pedro Nunes, Mercator and Escher" $\langle http://www.instituto-camoes.pt/cvc/ciencia_eng/e6.html\rangle$

"Projections for Navigators and Radio Operators"
$\langle http://www.progonos.com/furuti/MapProj/Normal/ProjNav/projNav.html\rangle$