Determining Exo-Planetary Orbits using Radial Velocity Measurements

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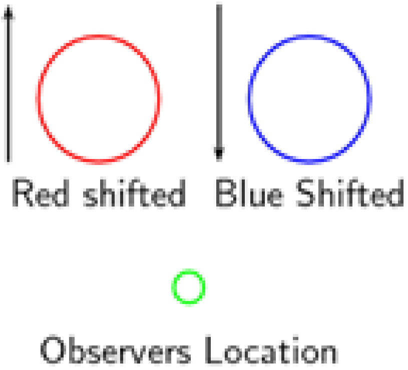
History of Radial Velocity Measurements

In 1952 Otto Struve proposed a method of using high powered telescopes to observe the slight displacement of a star created by a massive planet, such as Jupiter, orbiting it. Struve predicted that there would be a slight Doppler shift created by the star as the planet orbits it. Although at the time there were no instruments or techniques that would be able to detect such slight shifts in the electromagnetic spectrum. In the 1980's and early 1990's advances in spectral measuring devices enabled humans to observe slight variations in a stars Doppler signature. This technique is called Radial Velocity Measurement using Doppler Spectroscopy. In October 1995, 51 Pegasi b was the first exoplanet to be recorded and cataloged. Since then over 300 planets have been catalogued where many have been found using Doppler Spectroscopy.

What is a Doppler Shift

Doppler shifts work on the fact that as an object that is moving toward you has a higher recieved frequency as opposed to an object that is moving away from you.



Radial Velocity

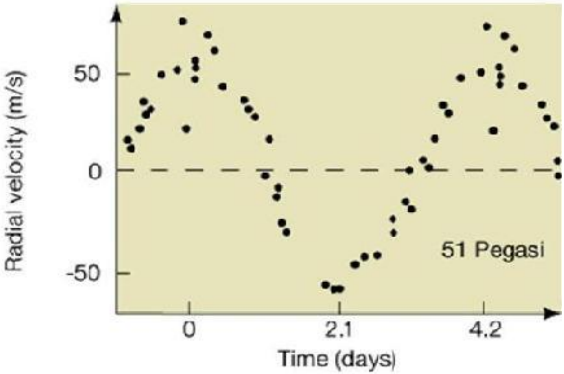
Using this equation we can determine the radial velocity of an object.

$$\frac{d\lambda}{dt} = \lambda_0(1 + v/c)$$

We solve for v here to find the radial velocity of the star, where $d\lambda/dt$ is the change in the doppler spectrum and λ_0 is the initial doppler value, and c is the speed of light.

51 Pegasi graph

If we plot time of one orbit versus the radial velocity we obtain a graph similar to this one



Kepler's Three Laws of Orbital Motion

In the mid 1600's Johannes Kepler devised these laws of planetary motion

1. planets travel in elliptical orbits with the parent star at one focus
2. planets sweep out equal areas in equal times
3. the ratio of the cube of the mean distance to the square of its orbital period is a constant.

Later in the 1700's Sir Issac Newton contributed these laws

1. a body in motion will remain in motion unless acted upon by an outside force, basically if a mass experiences a force it will experience an acceleration such that

$$F = ma$$

2. where F is the force,m is the mass, and a is the acceleration, lastly for every action there is an equal and opposite reaction.

$$\vec{F}_{12} = -\vec{F}_{21}$$

   Newton also contributed his universal law of gravitation which is

$$\vec{F} = \frac{Gm_1 m_2}{r^2}\hat{r}$$

Here G is the universal gravitation constant, m_1 and m_2 are the masses of the two objects, r is the distance between the two masses, and $\hat{r}$ is a unit vector that points in the direction of m_2 with its origin at m_1. By combining these equations we obtain

Converting to a Center of Mass frame of reference

Since the star's mass is creating a force on the planet, so to is the planet creating a force on the star. In turn both objects are actually orbiting a common center of mass. To find the effective mass of both objects at the center we obtain

$$\frac{1}{M_e} = \frac{1}{m_1} + \frac{1}{m_2}$$
$$M_e = \frac{m_1 m_2}{m_1 + m_2}$$
$$M_e = \frac{m_1 m_2}{M}$$

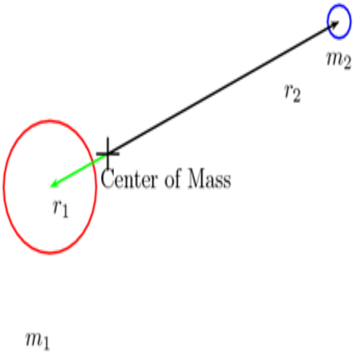
Also the ratio between the masses is

$$m_2 \left(1 + m_2/m_1\right)$$

keep these equations in mind as we will use them later.

Viewing the body system

Now that we have set up a different view of the system with a center of mass introduced both bodies move about it with thier own associated direction vectors r_1 and r_2.



Energy in Orbital Motion

To simplify the calculations for now we will assume the more massive object to be fixed in the center. We will use kinetic energy denoted K_E, and potential energy denoted P_E and note that both of these added together will give us the total energy E.

1. $K_E = 1/2 m_2 v_2^2$
2. $P_E = G m_1 m_2 / r$, which is gravitational potential energy

Putting all of these facts together we obtain

$$E = K_E + P_E$$

$$E = \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{r}$$

$$E = \frac{1}{2} m_2 \left(\frac{dx}{dt} + \frac{dy}{dt} \right)^2 - \frac{G m_1 m_2}{r}$$

A new cordiante system is nessasary

Since we now have dx's, dy's and r's we can simplify this by converting over to polar coordinates since both objects lie in a plane. We know that

1. $x = r\cos\theta$ and
2. $y = r\sin\theta$

It follows that

1. $\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$
2. $\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$

substituting back into our original equation for dx/dt and dy/dt we get

$$E = \frac{1}{2}m_2\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} + \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)^2 - \frac{Gm_1m_2}{r}$$

$$E = \frac{1}{2}m_2\left(\frac{dr}{dt} + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{Gm_1m_2}{r}$$

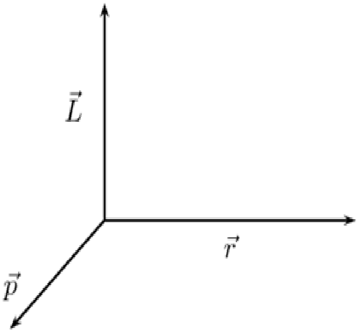
We will label this one and come back to it in a little bit

Deriving the Angular Momentum of the system

We will now look at the angular momentum of the system by the definition which is

$$\vec{L} = \vec{r} \times \vec{p}$$

. Graphically this looks something like this



Angular Momentum in Polar Coordinates

In the case of orbiting bodies the angular momentum is the position vector crossed with the velocity vector. But we want this to be in polar form to match our previous equation of the total energy.

$$
\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
L &= rp \sin\theta \\
L &= x\frac{dy}{dt} - y\frac{dx}{dt} \\
L &= r^2\frac{d\theta}{dt}
\end{aligned}
$$

This result is very familiar from the last equation so lets substitute it in to try to simplify our result for the total energy.

Substituting with Angular Momentum

Our last equation of the energy in the system was

$$E = \frac{1}{2}m_2\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{Gm_1m_2}{r}$$

If we now substitute in L for $r^2 d\theta/dt$ we get

$$E = \frac{1}{2}m_2\left(\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{r^2}\right) - \frac{Gm_1m_2}{r}$$

$$E = \frac{m_2\vec{L}^2}{2r^2} + \frac{m_2}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_1m_2}{r}$$

Now we can put this equation back into the reference frame with the center of mass and obtain

$$E = \frac{m_2 L^2}{2r^2} + \frac{m_2}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_1 m_2}{r}$$

$$E = \frac{1}{2} m_2 \left(1 + \frac{m_2}{m_1}\right) + \frac{dr}{dt} + \frac{1}{2}\frac{m_2 \left(1 + m_2/m_1\right) L^2}{\left(1 + m_2^2/m_1^2\right) r^2} - \frac{Gm_1 m_2}{\left(1 + m_2/m_1\right) r}$$

So now we have found a differential equation that relates the two orbiting bodies to a common center of mass and describes the orbital motion of both bodies. We are now able to correlate this to the radial velocity measurements taken by observing a stars motion over a certain interval of time. If E is greater than 1 the planet will fly off into space in a hyperbolic orbit, conversely if E is less than 1 the planet will crash into the star, but if E is zero it will remain in a constant orbit.

Distance Between the Two Bodies

To find the distance between the two orbiting bodies we can use Kepler's third law and obtain

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

$$GMT^2 = 4\pi^2 r^3$$

$$r^3 = GM \left(\frac{T}{2\pi} \right)^2$$

Since we know the period of the star's orbit T, and the star's radial velocity V_s we can determine the star's distance from the center of mass and denote it as R_s. Here $\omega = d\theta/dt$ or $\omega = 2\pi/T$

$$V_s = R_s \omega$$

$$R_s = \frac{V_s}{\omega}$$

The Planets Velocity

The planets velocity can be determined by seeing that the kinetic energy of the planet is created by the gravitational force exerted on it so they are equal

$$\frac{1}{2}mv_p^2 = \frac{GMm}{r^2}$$

$$v_p^2 = \frac{2GM}{r^2}$$

$$v_p = \sqrt{\frac{2GM}{r}}$$

Determining the Planet's Mass

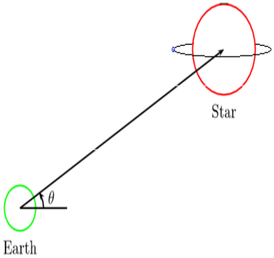
Now that we have the velocity of the planet and the star and we know the mass of the star we can determine the mass of the planet

$$M_p v_p = M_s V_s$$
$$M_p = \frac{M_s V_s}{v_p}$$

This is just an estimate of the planets mass though because due to the inclination of the observers view this number will change so we must adjust it by using this equation where K is the maximal radial velocity, v_s is the velocity of the star, and θ is the angle of inclination.

$$KM_s = v_s m_p \sin\theta$$
$$m_p \sin\theta = \frac{KM_s}{v_s}$$

Graphically this looks like



Now we have values for all the parameters in our equation so we can solve for E and have a complete picture of the two-body system.