Determining Exo-Planetary Orbits using Radial Velocity Measurements

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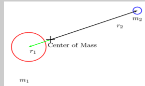
Abstract

The detection of extra solar planetary systems has become an active field of study through the use of viewing the displacement of a star due to a planet orbiting it, also known as radial velocity measurement. Using classical planetary motion and differential equations we can find out vital information about the planet such as mass and distance from the parent star. If the mass and the distance of the planet seem close to that of Earth we can then analyze it using other techniques to determine its atmosphere and the possibility for life on the planet.



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1. History

1.1. Clasical Planetary Motion

The study of planetary motion owes much of its creation to Johannes Kepler and Sir Issac Newton. Kepler contributed his three laws based on observational data which are;

1. planets travel in elliptical orbits with the parent star at one focus
2. planets sweep out equal areas in equal times
3. the ratio of the cube of the mean distance to the square of its orbital period is a constant.

Newton contributed his own three laws which are;

1. a body in motion will remain in motion unless acted upon by an outside force, basically if a mass experiences a force it will experience an acceleration such that

$$F = ma \tag{1}$$

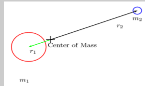
2. where F is the force,m is the mass, and a is the acceleration, lastly for every action there is an equal and opposite reaction.

$$\vec{F}_{12} = -\vec{F}_{21} \tag{2}$$

Newton also contributed his universal law of gravitation which is

$$\vec{F} = \frac{Gm_1 m_2}{r^2}\hat{r}. \tag{3}$$

Here G is the universal gravitation constant, m_1 and m_2 are the masses of the two objects, r is the distance between the two masses, and $\hat{r}$ is a unit vector that points in



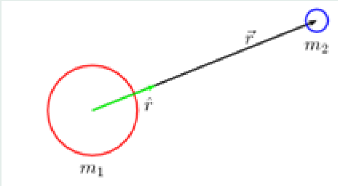


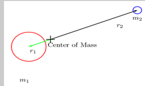
Figure 1: Where one objects center is the center of mass.

the direction of m_2 with its origin at m_1. Combining Equation (1), Equation (2), and Equation (3) and noting that a can also be seen as the second derivative of the position vector with respect to time.

$$\begin{aligned} m\vec{a} &= -\frac{Gm_1m_2}{r^2}\hat{r} \\ m\frac{d^2\vec{r}}{dt^2} &= -\frac{Gm_1m_2}{r^2}\hat{r} \end{aligned} \tag{4}$$

But Newton's second law states that for every action there is an equal and opposite reaction, which means that just as the star pulls on the planet the planet pulls on the star. So Figure 1 is not exactly correct because both objects will actually orbit around an unseen point in space due to their mutual gravitational attraction much like Figure 2.

Since both objects are in motion now this complicates our equations quite a bit. The first thing we must do is define what the center of mass is. We can denote M as the combined mass, so $M = m_1 + m_2$, we will also use the effective mass M_e which is



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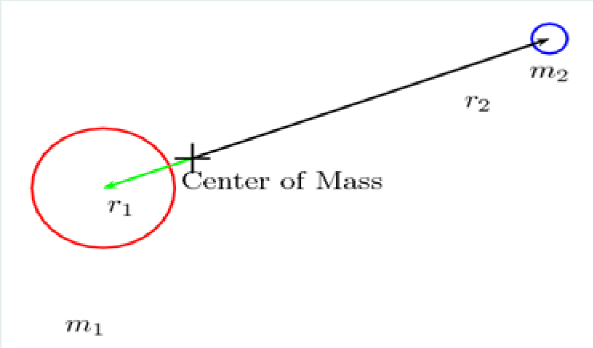


Figure 2: Where both objects are orbiting a common center of mass.

defined as

$$\begin{aligned} \frac{1}{M_e} &= \frac{1}{m_1} + \frac{1}{m_2} \\ M_e &= \frac{m_1 m_2}{m_1 + m_2} \\ M_e &= \frac{m_1 m_2}{M} \end{aligned} \tag{5}$$

We will keep Equation (5) in mind and return to this thought later. To obtain an equation for the orbital motion of both bodies we will begin by assuming the more massive object to be fixed and look at the total energy of the system which we will denote as E. We will use the fact that $E = K_E + P_E$ where K_E is the kinetic energy of the system and P_E is the potential energy of the system, which in this case is the gravitational potential energy. If we set $K_E = 1/2 m_2 v_2^2$ and $P_E = G m_1 m_2 / r$ then we arrive at the equation for E.

$$\begin{aligned} E &= \frac{1}{2}m_2 v_2^2 - \frac{Gm_1m_2}{r} \\ E &= \frac{1}{2}m_2\left(\frac{dx}{dt}^2 + \frac{dy}{dt}^2\right) - \frac{Gm_1m_2}{r} \end{aligned} \tag{6}$$

Here we can begin to see the trouble in that we have r, dx/dt, and dy/dt so we will remember Equation (6) and convert to polar coordinates, where everything will be in terms of r and θ.

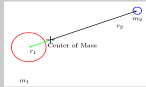
$$\begin{aligned} x =& r\cos\theta \\ \frac{dx}{dt} =& \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} \end{aligned}$$

and for the y-coordinate we obtain

$$\begin{aligned} y =& r\sin\theta \\ \frac{dy}{dt} =& \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt} \end{aligned}$$

Inputting these equivalencies back into Equation (6) we obtain

$$\begin{aligned} E &= \frac{1}{2}m_2\left(\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)^2\right) - \frac{Gm_1m_2}{r} \\ E &= \frac{1}{2}m_2\left(2\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right)^2 - \frac{Gm_1m_2}{r} \end{aligned} \tag{7}$$



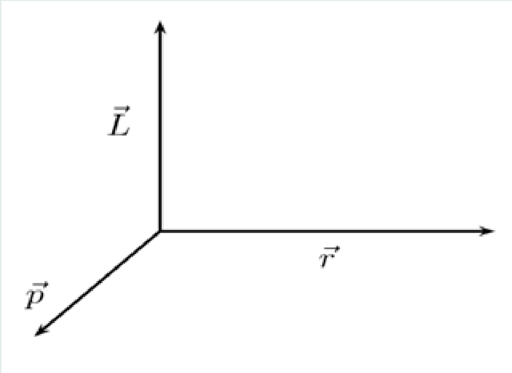
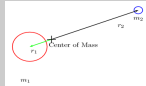


Figure 3: Angular Momentum.

1.2. Angular Momentum

To obtain a true picture of the orbital pattern of the two objects in is imperative that we introduce the concept of angular momentum. Angular momentum states that the effect on one object will be translated to the other object if no other external forces are involved. Simply stated, if one object speeds up the other will slow down, and if one slows down the other speeds up. This is the reason why in elliptic two body systems both objects will have the same eccentricities and when one is at shortest point from the focal point the other will be at its furthest. For now we are only looking at the orbiting body so we can simple write the angular momentum of the body L as $\vec{L} = \vec{r} \times \vec{p}$. Since the angular momentum is a cross product it is a vector perpendicular to the displacement vector $\vec{r}$ and the momentum vector $\vec{p}$, but $\vec{L}$ is a constant because if $\vec{r}$ becomes shorter the velocity vector will become larger and vice-versa.

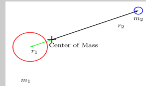


$$\vec{L} = \vec{r} \times \vec{p}$$
$$L = rp\sin\theta$$
$$L = x\frac{dy}{dt} - y\frac{dx}{dt}$$
$$L = r^2\frac{d\theta}{dt}$$

This result seems familar because the value $r^2 d\theta/dt$ appeared in Equation (7) so we can now substitute this value back into Equation (7) and notice that it is a scaling factor for the kinetic energy.

$$E = \frac{1}{2}m_2\left(2\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{Gm_1m_2}{r}$$
$$E = \frac{1}{2}m_2\left(2\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{r^2}\right) - \frac{Gm_1m_2}{r} \qquad (8)$$
$$E = \frac{m_2L^2}{2r^2} + m_2\left(\frac{dr}{dt}\right)^2 - \frac{Gm_1m_2}{r}$$

From Equation (8) we can see the orbital motion of an body if the center of mass is fixed, and a body with a lot more mass than the orbiting body. Equation (8) is a coupled first order differential equation because both θ and $\vec{r}$ are allowed to vary over a certain time interval. Also the total energy $\vec{E}$ and the angular momentum $\vec{L}$ are input parameters that not only define the orbit of the body but also the velocity at any instant in time which is dependent on the bodies distance from the other object. But what if the orbiting body has enough mass that it pulls on the star with enough force to make the star move and hence both objects will orbit a center of mass.



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1.3. Back to the Center of Mass

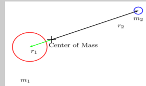
We have already found an equation that relates the center of mass and the mass of the bodies in orbit. At this time we would like to add the relation between the two bodies' masses and their distance from the center of mass. We can create the ratio $m_1\vec{r_1} = -m_2\vec{r_2}$, where the negative sign appears from Newton's second law. If we look back to Equation (5) and solve for m_2 we obtain $m_2\,(1 + m_2/m_1)$ which is a ratio of how the masses relate to each other. We can plug this result into Equation (8) to obtain an equation of orbital motion that relates both objects to the center of mass.

$$
\begin{aligned}
E &= \frac{m_2 L^2}{2r^2} + m_2\left(\frac{dr}{dt}\right)^2 - \frac{Gm_1m_2}{r} \\
E &= m_2\left(1 + \frac{m_2}{m_1}\right)\left(\frac{dr}{dt}\right)^2 + \frac{m_2\,(1 + m_2/m_1)\,L^2}{2\,(1 + m_2^2/m_1^2)\,r^2} - \frac{Gm_1m_2}{(1 + m_2/m_1)\,r}
\end{aligned}
\tag{9}
$$

So now we have found a differential equation that relates the two orbiting bodies to a common center of mass and describes the orbital motion of both bodies. We are now able to correlate this to the radial velocity measurements taken by observing a stars motion over a certain interval of time.

2. Radial Velocity and Doppler Shifts

In 1952 Otto Struve proposed a method of using high powered telescopes to observe the slight displacement of a star created by a massive planet, such as Jupiter, orbiting it. Struve predicted that there would be a slight Doppler shift, which is a change in the light spectrum caused when a light producing object moves towards or away from an observer, as seen in Figure 4. Although at the time there were no instruments or techniques that would be able to detect such slight shifts in the electromagnetic



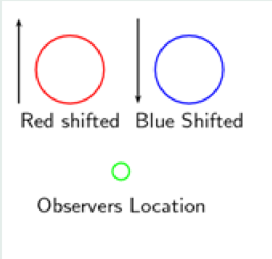


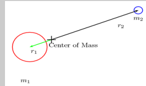
Figure 4: Doppler Shift in Light Waves

spectrum. In the 1980's and early 1990's advances in spectral measuring devices enabled humans to observe slight variations in a stars Doppler signature. This technique is called Radial Velocity Measurement using Doppler Spectroscopy. In October 1995, 51 Pegasi b was the first exoplanet to be recorded and cataloged. Since then over 300 planets have been catalogued where many have been found using Doppler Spectroscopy.

Doppler Spectroscopy works on the fact that a light emitting object that is moving away from us emits light on a red frequency and a light emitting object moving towards us emits light on a blue frequency. Therefore a star can be monitored over a certain time interval and the Doppler shift between red and blue frequencies can be measured. Now since we have an observed amount of Doppler shift, generally this number is less than 1, we can use the equation

$$\frac{d\lambda}{dt} = \lambda_0 \left(1 + \frac{V}{c}\right)$$

Where V is the radial velocity, c is the speed of light, $d\lambda/dt$ is the change in the doppler spectrum, and λ_0 is the initial doppler value. Next by plotting the stars period versus



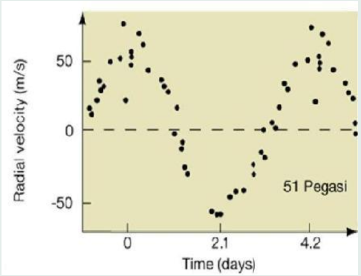
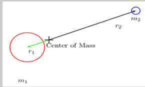


Figure 5: Radial Velocity graph of 51 Pegasi b

its radial velocity we get a graph like Figure 5, which is a graph of 51 Pegasi. 51 Pegasi is the first star to have a known planet orbiting it, the graph was formed by taking repeated measurements of the Doppler shifts that the star has. From this graph we can see the period of the planet because that will be the same as the period of one oscillation of the star's period. For a circular orbit this graph looks very much like a sine curve or an undamped forced harmonic oscillator which makes sense as the only forces are between the planet and the star and it is forced because of the centripetal acceleration on the planet which is why it is in orbit and not crashing into the star or being flung off into space. If the planet has a more eccentric orbit this graph looks quite a bit different. We also know the mass by noting its luminosity versus its distance along with the star's chemical structure and temperature.

2.1. The Distance from the Center of Mass to the Planet

Since we know the period of the star's orbit T, and the star's radial velocity V_s we can determine the star's distance from the center of mass and denote it as R_s. Here



$\omega = d\theta/dt$ or $\omega = 2\pi/T$

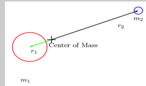
$$
\begin{aligned}
V_s &= R_s\omega \\
R_s &= \frac{V_s}{\omega} \\
R_s &= \frac{V_s T}{2\pi}
\end{aligned}
$$

We know that both the planet and the star have the same period of orbit so we can denote the period of the planet as T and the mass of the star as M, we can determine the distance the planet is from the star as r. Using Kepler's third law we obtain

$$
\begin{aligned}
T^2 &= \frac{4\pi^2 r^3}{GM} \\
GMT^2 &= 4\pi^2 r^3 \\
r^3 &= GM\left(\frac{T}{2\pi}\right)^2
\end{aligned}
$$

So the distance from the planet to the center of mass is just $r - R_s = R_p$. Since we have a value for the distance the planet is from the star we can now determine the velocity of the planet by noting that the planets velocity is related to the star it is orbiting we can use rotational kinetic energy and see that it must be a constant, so the amount of force needed to keep the planet in orbit at an average velocity is

$$
\begin{aligned}
\frac{1}{2}mv_p^2 &= \frac{GMm}{r^2} \\
v_p^2 &= \frac{2GM}{r^2} \qquad (10) \\
v_p &= \sqrt{\frac{2GM}{r}}
\end{aligned}
$$



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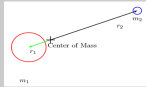
In Equation (10) v_p is the velocity of the planet, M is the mass of the star, and r is the distance from the center of mass to the planet. Although this is an average velocity for the planet, depending on the eccentricity and the observed radial velocity of the star using Kepler's second law an instantaneous velocity for the planet can be found in a similar manner. In the simple case where the planet is in a circular orbit the planet orbits the center of mass at a distance r with a constant velocity v_p. A simple ratio between the planet and star reveals

$$\begin{aligned} M_p v_p &= M_s V_s \\ M_p &= \frac{M_s V_s}{v_p} \end{aligned} \tag{11}$$

Although the value for the mass of the planet, M_p, is just an estimate because this all depends on the angle of inclination of the planetary orbit of the obeserved star from the position of the observer. The actual mass of the planet is adjusted by Equation 12 where θ is the angle of inclination of the observed orbital plane and the observer and K is the maximal radial velocity and v_s is the velocity of the star.

$$\begin{aligned} K M_s &= v_s m_p \sin\theta \\ m_p \sin\theta &= \frac{K M_s}{v_s} \end{aligned} \tag{12}$$

The minimum mass of the planet is $m_p \sin\phi$ which is measured multiples of Jupiter's mass as Jupiter sized planets create the most amount of stellar wobble. This is can be seen in Figure 6



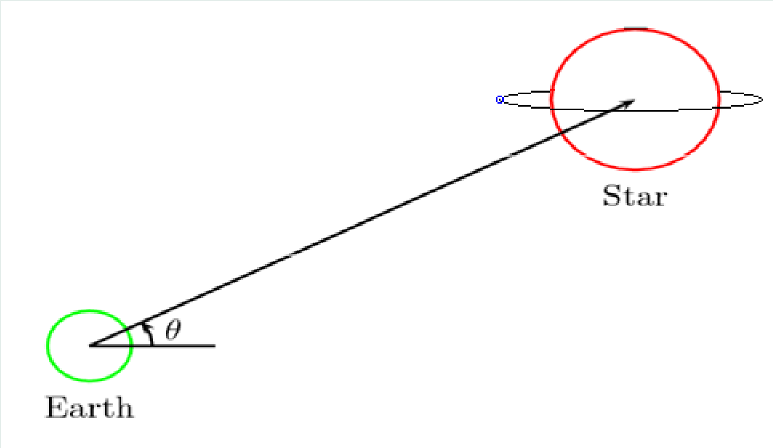
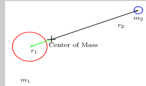


Figure 6: Inclination of Stellar Object When Viewed From Earth



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3. Conclusion

Now that we have found the mass of the planet we can use the mass of the star, m_1, the mass of the planet m_2 as parameters to find the angular momentum, L, which will give us the total orbital energy E. Using m_1, m_2, L, and E as parameters in our differential equation of orbital motion, Equation (9) we can obtain the entire motion of the two body system. This system is all determined initially from the radial velocity measurement that was taken from observing a star over time. This process has been used to determine many other extra solar systems and has ushered in a new era in planetary searches. Although new methods are being implemented to find smaller planets this process is still used to observe systems that may be close to our own solar system with large planets in large almost circular orbits where a possibility of a habitable planet closer to the star is possible. This same idea can be expanded into an n-body system but the equations relating the bodies become very complex very rapidly. Also switching to polar coordinates would not work because all the bodies may not be orbiting in a plane so a better coordinate system must be adopted such as a spherical coordinate system.

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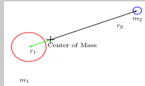

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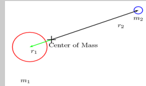

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