# Bungee Parachute

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**Abstract**

This paper has undergone a series of modifications; originally it was to use a yoyo to store up energy so that a soft landing might be obtained from a parachute without using a ridiculously large canopy. The yoyo however has a number of issues, starting with the difficulty of calculating the motion of a yoyo with varying drum diameter, which is more or less necessary for the concept. The yoyo would also have to be heavy and bulky in this application, not to mention that it would never give a smooth transition between moving down and moving up. Therefore a spring, or more specifically a bungee chord will take the place of the yoyo. Bungee chords have a nice smooth transition between decelerating a body in the down ward direction and reaccelerating it in the upward direction. They are reliable and effective for the purpose at hand; as proven by the various groups of people who seem to enjoy jumping off of perfectly good bridges, with one tied to their ankles. Bungees are also reasonably light and convenient for storage.

The concept for this paper is that if a mass parachute system is at a stable terminal velocity descent and the mass is allowed to go into free fall until it's motion is effected by a bungee chord that is attached to the mass. The parachute will provide an opposing force that the mass will work against resulting in the stretching of the bungee chord. This stretching will slow until the mass has achieved maximum extension of the spring mass system. At this point the mass will change direction and begin to move back towards the parachute due to the restoring force of the spring/bungee. The goal is to have the mass at a velocity that is opposite and equal to the downward descent of the the total parachute system so at the instant the mass touches down it will be at a zero velocity.

For mathematical simplicity the initial condition of the system will have the bungee tight but unstretched with the mass supported by a line parallel to it.

## 1 Calculations for a Bungee

We need to start by defining our axis and variables

$$\sum F = ma$$

$$ma = mg - ky$$

Because $a = y''$, we have

$$my'' = mg - ky. \tag{1}$$

Now, when the mass is hanging at equilibrium, its position is $y_0$, so

$$0 = mg - ky_0,$$

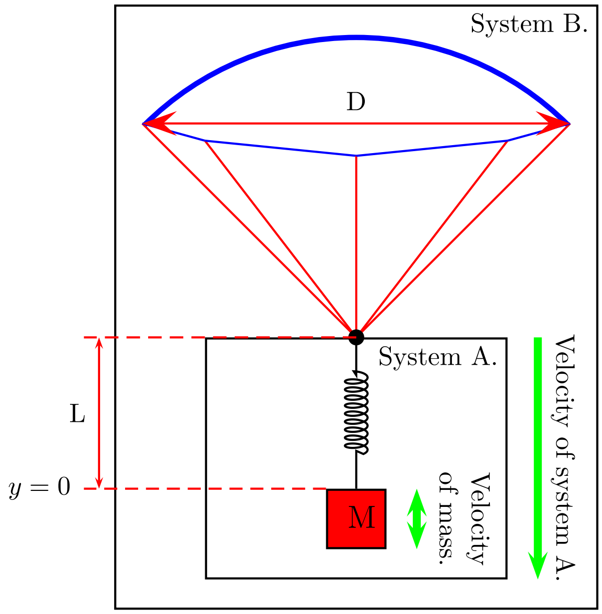


Figure 1: A Para-bungee

or equivalently,

$$mg = ky_0.$$

Substituting this result in equation (1), we have

$$\begin{aligned} my'' &= ky_0 - ky \\ my'' &= k(y_0 - y) \end{aligned}$$

Now we need to convert the system from units of $y$ to units of $z$. This will be done primarily using Figure 2 which gives us,

$$\begin{aligned} z &= y - y_0 \\ z' &= y' \\ z'' &= y'' \end{aligned} \tag{2}$$

Substituting these results into $my'' = k(y_0 - y)$, we get

$$mz'' = -kz. \tag{3}$$

Now that we have converted from y to z units with equation (2) it is time to start calculating the motion of the spring,

$$mz'' + kz = 0 \tag{4}$$

If we let

$$z = e^{\lambda t}, \tag{5}$$

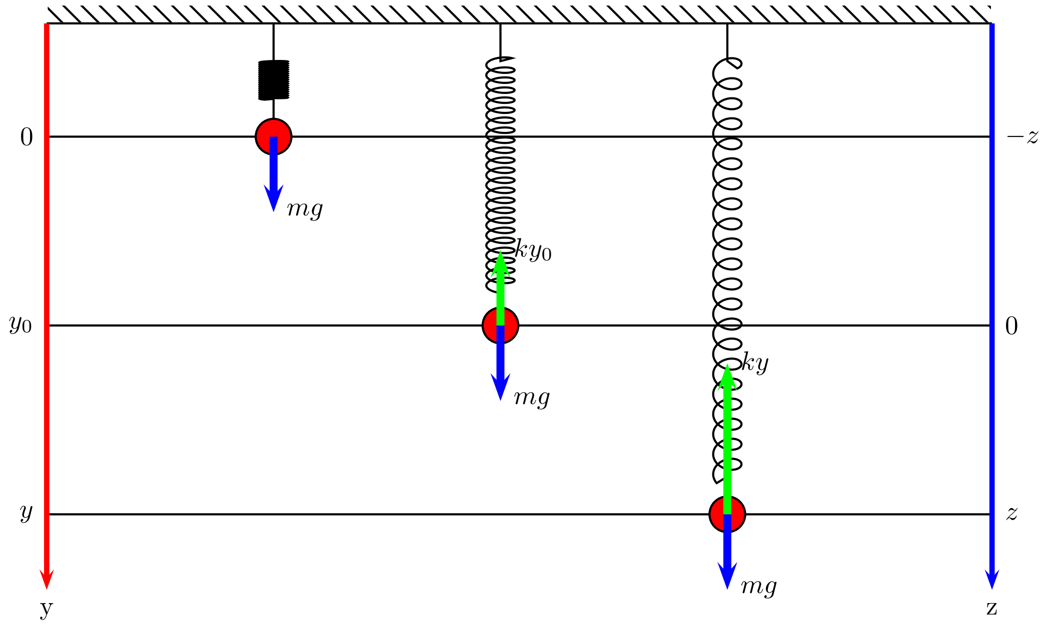

Figure 2: A set of springs.

then

$$z = e^{\lambda t}$$
$$z' = \lambda e^{\lambda t}$$
$$z'' = \lambda^2 e^{\lambda t}.$$

Plug these results into

$$mz'' + kz = 0,$$

and solve for $\lambda$.

$$m\lambda^2 e^{\lambda t} + ke^{\lambda t} = 0$$
$$m\lambda^2 + k = 0$$
$$\lambda^2 = -\frac{k}{m}$$
$$\lambda = \pm i\sqrt{k/m}$$

If we put $\lambda$ back in to equation (5), then we get the complex solution

$$w = e^{it\sqrt{k/m}(it)}$$
$$w = \cos\left(\sqrt{k/m} * t\right) + i \sin\left(\sqrt{k/m} * t\right).$$

The real and imaginary parts of the solution are both solutions so the general solution is

$$z = A\cos\left(\sqrt{k/m} * t\right) + B\sin\left(\sqrt{k/m} * t\right). \tag{6}$$

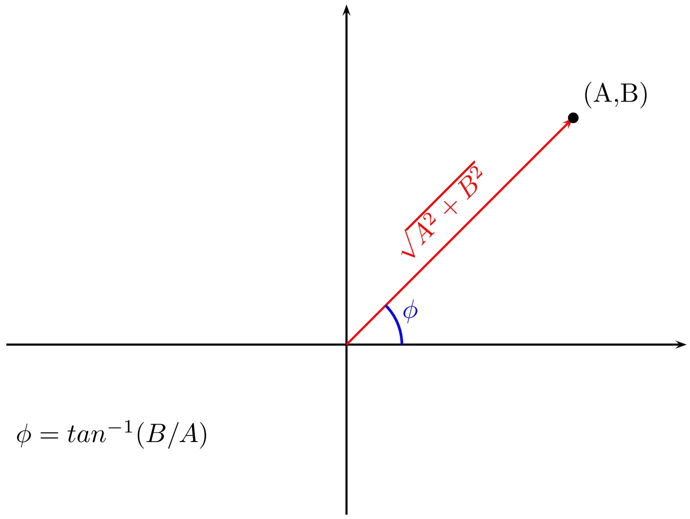


Figure 3: Defining $\phi$.

As per Figure 3 we will look at the coefficients $A$ and $B$, and use $A$ and $B$ to define a vector. Multiplying and dividing the whole equation by the calculated length of that vector we have,

$$z = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos\left(\sqrt{k/m}*t\right) + \frac{B}{\sqrt{A^2+B^2}}\sin\left(\sqrt{k/m}*t\right)\right]. \tag{7}$$

Since $A$ divided by the length of the vector defined by $A$ and $B$ is the $cos(\phi)$ and $B$ divided by the length of the vector is $\sin(\phi)$,

$$z = \sqrt{A^2+B^2}\left[\cos(\phi)\cos\left(\sqrt{k/m}*t\right) + \sin(\phi)\sin\left(\sqrt{k/m}*t\right)\right]$$

which simplifies to,

$$z = \sqrt{A^2+B^2}\left[\cos\left(\sqrt{k/m}*t-\phi\right)\right].$$

Finally we can integrate the result to get,

$$z' = -\sqrt{A^2+B^2}\sqrt{k/m}\left[\sin\left(\sqrt{k/m}*t-\phi\right)\right].$$

Now that we have $z$ and $z'$ it is time to solve for our initial conditions, $A$, $B$, and $\phi$.

$$z = A\cos\left(\sqrt{k/m}*t\right) + B\sin\left(\sqrt{k/m}*t\right)$$

Since we are starting with an unstreched spring and basing our equations off of an equilibrium position the initial condition may be defined by,

$$\begin{aligned}
F = -kz(0) &= mg \\
z(0) &= -\frac{mg}{k} \\
-\frac{mg}{k} &= A\cos\left(\sqrt{k/m}*0\right) + B\sin\left(\sqrt{k/m}*0\right) \\
-\frac{mg}{k} &= A.
\end{aligned}$$

We can now use this initial condition for $z(0)$.

$$\begin{aligned}
z &= A\cos\left(\sqrt{k/m}*t\right) + B\sin\left(\sqrt{k/m}*t\right) \\
-\frac{mg}{k} &= A\cos\left(\sqrt{k/m}*0\right) + B\sin\left(\sqrt{k/m}*0\right) \\
A &= -\frac{mg}{k}
\end{aligned}$$

Since the initial velocity of the mass in the frame of reference of system A in figure (1) is zero,

$$z'(0) = 0.$$

Now we can calculate the values of $A$ and $B$,

$$\begin{aligned}
z' &= A\sqrt{k/m}\sin\left(\sqrt{k/m}*t\right) + B\sqrt{k/m}\cos\left(\sqrt{k/m}*t\right) \\
0 &= -A\sqrt{k/m}\sin\left(\sqrt{k/m}*0\right) + B\sqrt{k/m}\cos\left(\sqrt{k/m}*0\right) \\
0 &= B\sqrt{k/m} \\
B &= 0
\end{aligned}$$

Since $B$ is equal to zero and $\phi$ is equal to $\tan^{-1}(B/A)$,

$$\begin{aligned}
z &= -\frac{mg}{k}\cos\left(\sqrt{k/m}*t-0\right) \\
z &= -\frac{mg}{k}\cos\left(\sqrt{k/m}*t\right),
\end{aligned}$$

and,

$$\begin{aligned}
z' &= \frac{mg}{k}\sqrt{k/m}\sin\left(\sqrt{k/m}*t-0\right) \\
z' &= g\sqrt{\frac{m}{k}}\sin\left(\sqrt{k/m}*t\right).
\end{aligned}$$

## 2 Calculations for the parachute.

System B will be defined by

$$v = 29 \times \sqrt{\frac{F}{1.4 \times \pi r^2}} \tag{8}$$

The equation (8) is the contribution of Reference [1] from which it was directly copied.

The force will be defined by,

$$F = k(z - \frac{mg}{k}$$
$$= kz - mg.$$

Entering this in to equation (8) for F gives us

$$v = 29 \times \sqrt{\frac{kz - mg}{1.4 \times \pi r^2}}.$$

Where $z$ is defined by,

$$z = -\frac{mg}{k} \cos\left(\sqrt{k/m} * t\right).$$

# 3 Conclusion

In this paper we have generated three equations needed to model this system they are,

$$z = -\frac{mg}{k} \cos\left(\sqrt{k/m} * t\right).$$
$$v = 29 \times \sqrt{\frac{kz - mg}{1.4 \times \pi r^2}}.$$
$$z' = g\sqrt{\frac{m}{k}} \sin\left(\sqrt{k/m} * t\right)$$

I took these results and wrote a MATLAB m-function that uses them to generate 'useful' results. These outputs should be taken with a grain of salt because I have yet to make a physical test that definitively agreed with them, however my preliminary tests with a two foot diameter paper parachute, a rubber band with spring constant $k \approx 0.25$ and a 1/8lb load seem to be close to agreeing with my pre-calculated results. The physical system seems to differ primarily in that the dampening forces of the rubber band are much larger than I anticipated for 3/4 of a period when I neglected them in my calculations. Here is the MATLAB input and output for that system:

We provided the following input into our program:

- The load in lbs = .125 .
- The stretch in lbs per foot applied to the bungee = .2 .
- The diameter of the parachute in feet = 2 .

The following are results returned by the program.

- The release height is 3.91 feet off of the ground.
- Landing velocity feet/sec if bungee had not been deployed 4.89 .
- Landing velocity in feet/sec with bungee deployment 0.42 .

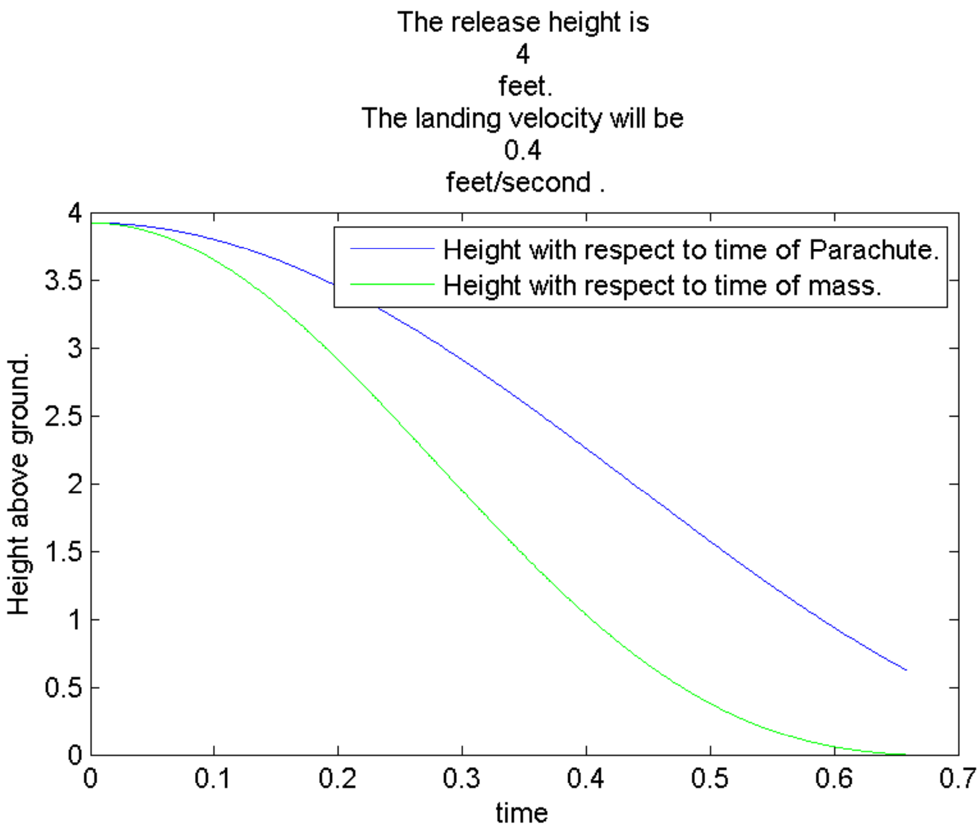


Figure 4: This figure shows the position with respect to time of both the parachute and the mass with a .125 lb load and a 2foot parachute.

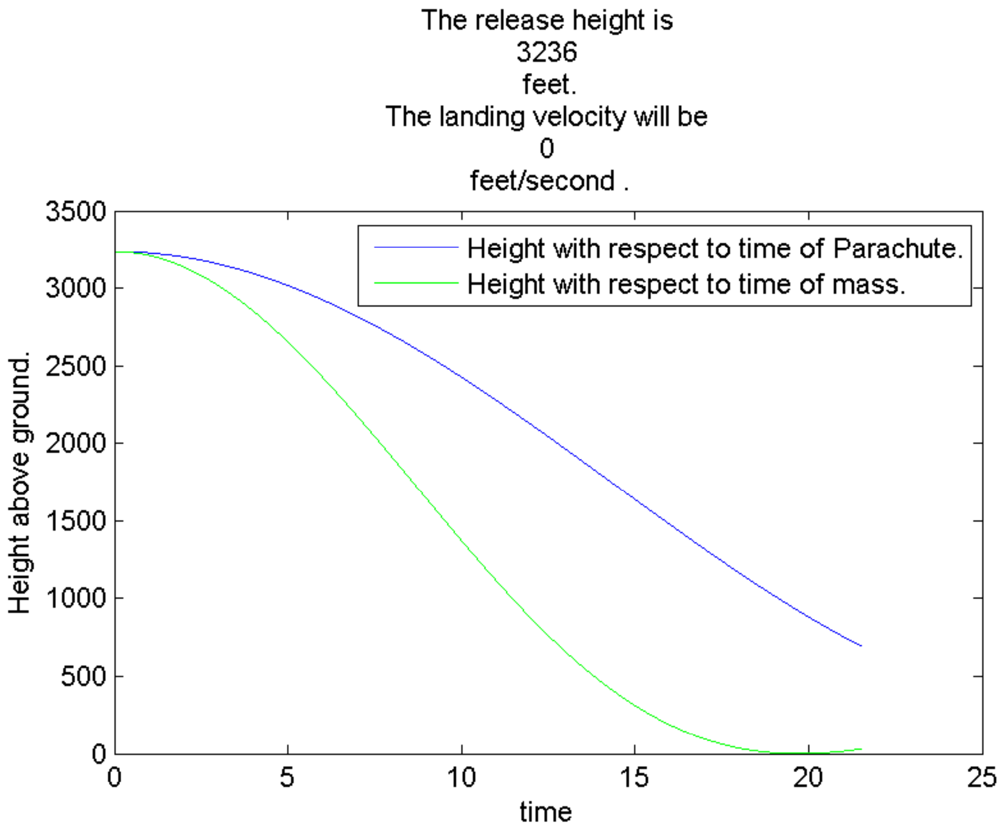


Figure 5: This figure shows the position with respect to time of both the parachute and the mass with a 10,000lb load and 24 foot parachute.

Figure 4 represents the results of the above block, showing the position with respect to time of both the parachute and the mass with reference to the height of release.

Now to show the real benefits of this idea I will run the program again except with a load of 10,000lbs and a 24 foot diameter parachute, note that this is approximately the size of an emergency chute intended for landing people alive but not uninjured. To make this work as well as possible I have through trial and error determined that a spring constant of 15lbs/foot is appropriate.

We provided the following input into our program:

- The load in lbs = 10000 .
- The stretch in lbs per foot applied to the bungee = 15 .
- The diameter of the parachute in feet = 24 .

The following are results returned by the program.

- The release height is 3227.27 feet off of the ground.

- Warning, bungee must be released immediately upon ground contact.
- Landing velocity feet/sec if bungee had not been deployed 115.23 .
- Landing velocity in feet/sec with bungee deployment 4.23 .

As you can see in Figure 5 the use of a bungee in the system makes the difference between an extremely hard landing and a rather soft one. I should note that for this system it may be necessary to take in to account the air drag upon the load (which I have neglected) depending upon the aerodynamic characteristics of the load

The displayed Warning simply alerts us to the fact that after touching the ground the bungee will draw the load back up off the ground which may result in an unpleasantly hard second landing.

According to the equations above it is possible to land a man safely with only a ten foot parachute; the advantages to this should be obvious, smaller lighter parachutes and faster descents. This would also be use full for landing scientific probes on other planets - no more weird shock absorbing landing balls or landing rockets. Of course the air density would differ from here on earth as would the gravitational constant.

For improved performance the object should initially be up attached to the shroud lines with the bungee draped below, allowing a free fall prior to stretching the bungee, this would allow more energy storage in a smaller bungee. Another idea that would facilitate a safe landing and good reliability would be to hang a weight of substantial mass a calibrated distance below the 'delicate' load and attached to the same end of the bungee. The whole system would then reach a stable velocity as described by equation 8 until the lower mass contacts the ground, then the stored stretch in the bungee will begin accelerating the 'important' load back up, until hopefully it reaches the ground with a near zero vertical velocity.

## References

[1] Fluid-Dynamic Drag, Dr.-Ing. Sighard F. Hoerner