

# Free Falling Objects and Reynolds Numbers

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## Abstract

This paper will discuss the underlying physics of the forces acting on a free-falling object, and the drag forces acting on those objects as well as

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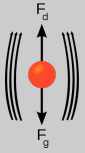
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Reynolds Numbers. From this knowledge, we will construct and derive a differential equation to more properly explain the motion of falling objects of differing Reynolds Numbers and attempt to explain the vast error associated with neglecting the effects of said Reynolds numbers.

## 1. Introduction

A classic problem given to beginning physics students involves a single particle of mass caught in a free-fall, being affected only by gravity, and is most often solved by assuming a constant acceleration of  $9.8 \text{ m/s}^2$  due to said gravity. Though, as the students inevitably move on in physics and mathematics, it becomes apparent that the reality of the situation is in fact much more complex. Clearly, it simply isn't logical to think that an object of discernable mass in a free-fall could sustain a constant acceleration indefinitely. This is due to air resistance, otherwise known as "drag". This paper shall attempt to explain said air resistance, and use basic knowledge to help formulate a more accurate differential equation to model the true effects of drag on an item of mass in free-fall.

## 2. The Basic Model for Motion

Before jumping into deriving our differential equation, I will quickly summarize and formulate our basic equations based on what we should already know about simple particle motion. The mathematical model for motion is found in Newton's second law:

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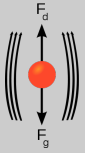
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$$F = ma$$

Where  $m$  corresponds to the mass,  $a$  corresponds to the acceleration, and  $F$  equates to the sum of all forces acting on the object. From our knowledge of basic calculus and physics, we should also know that the equation for the acceleration of an object is the derivative of the velocity equation for said object, and in the same way the velocity equation is the derivative of the position equation. We shall use the following relationships to our advantage in understanding and solving our ordinary differential equation:

$$a = \frac{dv}{dt} \quad (1)$$

$$v = \frac{dx}{dt} \quad (2)$$

Before taking air resistance into account, the  $F$  in Newton's second equation was usually denoted simply as the Force of Gravity. But for our purposes, we will separate the forces as such:

$$F_g + F_d = ma$$

In which  $F_g$  corresponds to the force of gravity and  $F_d$  corresponds to the drag force from air resistance. Generally, the drag force on an object will be considered proportional to the velocity according to the equation  $F_d = -kv$ .

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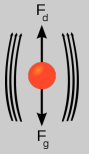
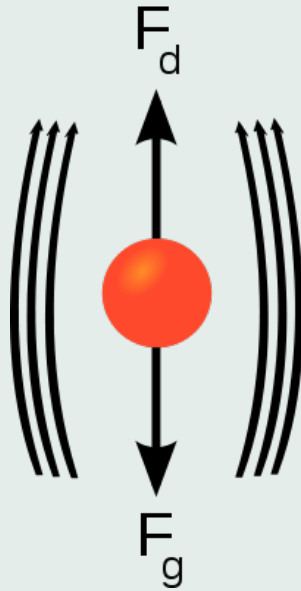
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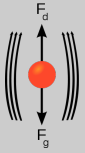
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$$F_g + F_d = ma \quad (3)$$

$$-mg - kv = ma \quad (4)$$

### 3. Reynolds Numbers

It is at this point that we may introduce the concept of the Reynolds number. Derived from the Navier-Stokes equations, the Reynolds number is a fluid dynamics principle that describes the ratio of the inertial and viscous forces that will act on an object of a characteristic size:

$$R = \frac{PDV}{u}$$

Where

$P$  = Fluid Density

$D$  = Characteristic Object Length

$V$  = Fluid Velocity

$u$  = Fluid Viscosity

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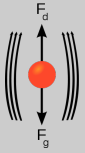
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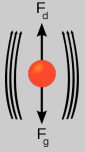
Essentially, the Reynolds number describes the relative importance of the forces in terms of their effects on the falling object. Traditionally, mathematicians and engineers have approximated the drag force on any object to be  $F_d = kv$  for all ranges of velocity. Unfortunately, this turns out to only be true for objects with a Reynolds number less than 1. For any Reynolds number greater than 1, the velocity dependence on the drag force will become exponential rather than linear. It is the relative size of the Reynolds Number value that determines the size of the exponent  $n$  for the drag force. In other words, for varying Reynolds numbers, the force of drag equation is written as such:

$$F_d = kv^n$$

While a Reynolds number less than 1 keeps a linear relationship, a Reynolds number above  $10^3$  may give as high as a quadratic relationship. Realistic Reynolds numbers may range from a value of 1 for a dust particle moving through air to a value of more than  $10^8$  for a submarine sinking in water. This relationship slightly alters equation 4 into the form:

$$-mg - kv^n = ma \quad (5)$$

As one can see, this new velocity dependence will greatly change the behavior of an object in free-fall. While the true exponent value for  $n$  may require a wind-tunnel test to calculate, we may still construct our differential equation and use Matlab experiment with differing  $n$  values.



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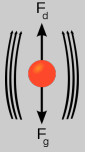
## 4. Building the Differential Equation

At this point, we have the option of building our DE in either the first order or second order. In the name of simplicity, we shall build and solve the equation in the first order. Using the relationships and equations from the last section, we will now build our DE. Knowing that  $F_d = -kv^n$ ,  $F_g = -mg$ , and  $a = \frac{dv}{dt}$ , we may rearrange our previous equation into the following form:

$$\begin{aligned} -mg - kv^n &= ma \\ -mg - kv^n &= m \frac{dv}{dt} \end{aligned}$$

We shall also assume that  $v(0) = 0$  for our equation. Rearranging our variables and dividing through by  $m$ , we will put the equation into the form  $dv/dt + p(t)v = q(t)$  as

$$\frac{dv}{dt} + \frac{k}{m}v^n = -g$$



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## 5. Solving the Differential Equation

Now that we have our basic equation in our desired form, we may notice that the value of  $n$  will have a great impact on our ability to solve and analyze the problem at hand. To give a few examples for arbitrary values of  $n$ ;

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

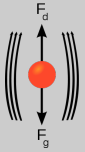
$$\frac{dv}{dt} + \frac{k}{m}v^2 = -g$$

$$\frac{dv}{dt} + \frac{k}{m}v^3 = -g$$

$$\frac{dv}{dt} + \frac{k}{m}v^4 = -g$$

As one can see, our new differential equation's complexity revolves entirely around the value of  $n$ . When the value of  $n$  equals 1, the problem is linear and only requires a simple integration factor to solve. But when the value of  $n$  exceeds the value of 1, the problem becomes nonlinear and in turn often becomes much more difficult to solve, if not impossible. To give definition as to the analysis that is to follow; it is our aim to demonstrate the effect of differing values of  $n$  on the maximum speed of the free-falling object, otherwise known as the object's "Terminal Velocity". In this way, we may show the vast error when applying the linear equation to an object with a very large reynolds number.

We will now work through and solve the simple linear version of our differential equation, where  $n = 1$ .



$$\frac{dv}{dt} + \frac{k}{m}v^1 = -g$$
$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

To begin, we may solve the DE by attempting to find an integrating factor  $\mu$ . This is done as such:

$$\mu = e^{\int p(t)dt}$$
$$\mu = e^{\int (k/m)dt}$$
$$\mu = e^{kt/m}$$

Now that we have the integrating factor, we may multiply our DE by  $\mu$ , and then proceed to integrate both sides of the equation.

$$e^{kt/m} \frac{dv}{dt} + e^{kt/m} \frac{k}{m}v = -ge^{kt/m}$$
$$[e^{kt/m}v]' = -ge^{kt/m}$$
$$e^{kt/m}v = - \int ge^{kt/m} dt$$
$$e^{kt/m}v = -\frac{mg}{k}e^{kt/m} + C$$

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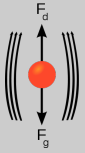
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At this point, we may use the initial condition of  $v(0) = 0$  to solve for  $C$ .

$$\begin{aligned}e^{k(0)/m}(0) &= \frac{-mg}{k}e^{k(0)/m} + C \\e^{(0)}(0) &= \frac{-mg}{k}e^{(0)} + C \\0 &= \frac{-mg}{k}(1) + C \\C &= \frac{mg}{k}\end{aligned}$$

Now that we have our  $C$  value, we may plug it back into our equation, divide through by  $e^{kt/m}$ , and with a little rearrangement, we will have our solved equation for the case of  $n = 1$ :

$$\begin{aligned}e^{kt/m}v &= \frac{-mg}{k}e^{kt/m} + C \\v &= \frac{\frac{-mg}{k}e^{kt/m} + \left(\frac{mg}{k}\right)}{e^{kt/m}} \\v &= \frac{-mg}{k} + \frac{mg}{k}e^{-kt/m} \\v &= \frac{mg}{k}(e^{-kt/m} - 1)\end{aligned}$$

To check our work, we may use a computer program such as Matlab to compute the symbolic answer to our differential equation. We may enter the following

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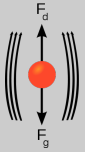
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command to solve for this version of our equation:

```
v_1=dsolve('Dv=-g-(k/m)*(v^1)', 'v(0)=0')
```

Which should give us

```
v_1=- (g*m/k)+exp(-m*t/k)*(g*m/k)
```

Which appears to be exactly what we solved before:

$$v_1 = -\frac{gm}{k} + e^{-kt/m}\frac{gm}{k}$$
$$v_1 = \frac{mg}{k}(e^{-kt/m} - 1)$$

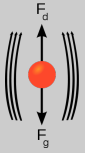
While the first case with  $n = 1$  was tedious, yet simple, the nonlinear versions of our equations may prove to be much more difficult. Using Matlab, we may use the following command to save us large amounts of time and to solve our equation for  $n = 2$ :

```
v_2=dsolve('Dv=-g-(k/m)*(v^2)', 'v(0)=0')
```

Which should produce the result

```
v_2=-tan((g*k*m)^(1/2)*t/k)*(g*k*m)^(1/2)/m
```

Which gives us the solved equation



$$v_2 = -\tan\left(\frac{t}{k}(mgk)^{\frac{1}{2}}\right)\frac{(mgk)^{\frac{1}{2}}}{m} \quad (6)$$

Though it may be occasionally time consuming to correctly decipher the solution, computer programs such as Matlab do wonders in simplifying the solving process. Of course, not all values of  $n$  will be solveable with Matlab, or at all. We could spend a great deal of time filling these pages with successful Matlab-solved answers for differing  $n$  values, but instead we shall move on to demonstrate the effects of said  $n$  values on the object's terminal velocity.

## 6. Testing the Results

Now that we understand what we need to look for, we shall quickly solve for our terminal velocity equation for all values of  $n$ . To do this easily, we shall bring back our generic  $n$ -equation and solve for  $v$  when  $a = 0$ :

$$\begin{aligned} -mg - kv^n &= ma \\ -mg - kv^n &= m(0) \\ -kv^n &= (0) + mg \\ v^n &= -\frac{mg}{k} \\ v &= \sqrt[n]{-\frac{mg}{k}} \end{aligned}$$

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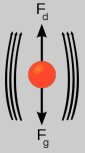
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While we may be put off by the negative sign at first glance, we must remember that we have the value for  $g$  set at  $-9.8$ , so we need not worry. That aside, we can now see the true impact of choosing an incorrect linear or nonlinear relationship on terminal velocity:

$$v_n = \sqrt[n]{-\frac{mg}{k}} \quad (7)$$

$$v_1 = -\frac{mg}{k} \quad (8)$$

$$v_2 = \sqrt[2]{-\frac{mg}{k}} \quad (9)$$

$$v_3 = \sqrt[3]{-\frac{mg}{k}} \quad (10)$$

$$v_4 = \sqrt[4]{-\frac{mg}{k}} \quad (11)$$

That is, the difference between the terminal velocity of an object with an  $n$  value of 1 to an object with an  $n$  value of 2 is that it is the square of the value at  $/mu = 2$ . To demonstrate these vast differences, and to check our terminal velocity equation, we shall graph our equations with static  $g, m, k$  values but at different  $n$  values using dfield7 for Matlab. For  $g = -9.8 \text{ m/s}^2$ ,  $m = 50 \text{ N}$ , and  $k = 0.5 \text{ N/s}$ :

As expected according to formula 9, the terminal velocity for  $n = 1$  appears to level out at 1960 in Figure 1. Now, let us compare that with  $n = 2$ . The graph in Figure 2 also abides by it's formula, found in equation 10. The terminal

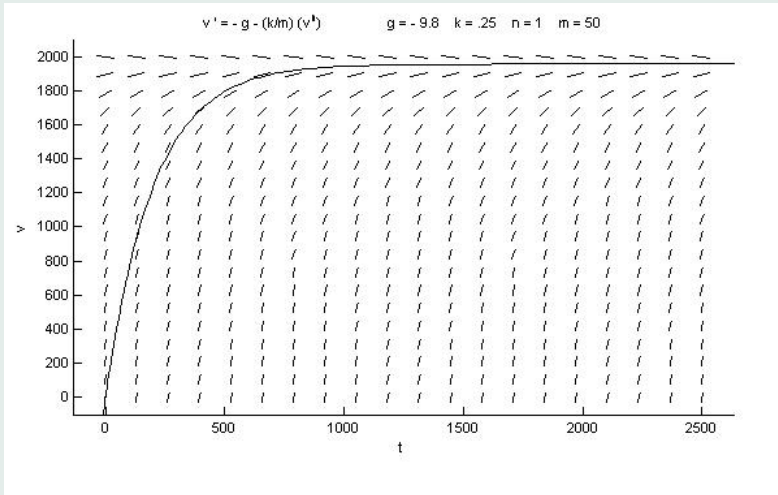
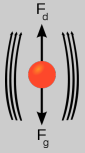


Figure 1:  $n = 1$ , Terminal Velocity Approx = 1960

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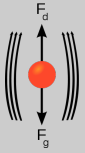
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velocity levels out around 44, which is nearly the square root of the preceding terminal velocity. The graph in Figure 3 also follows equation 11 as well, and the terminal velocity of 12.5 is cube-root of the initial terminal velocity. Lastly, we see our final graph in Figure 4 also abiding by its formula found in equation 12. Topping out around 6.6, it indeed shares the same relationship to the initial terminal velocity.

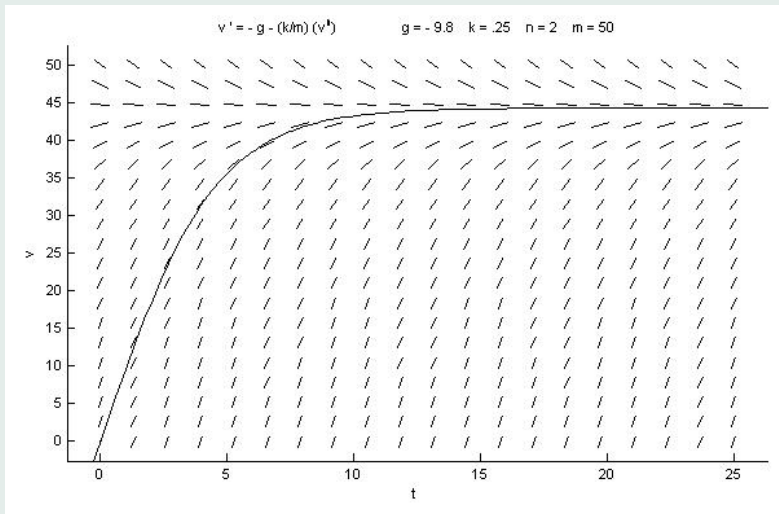


Figure 2:  $n = 2$ , Terminal Velocity Approx = 44

From the previous figures, anyone can clearly see the vast differences in

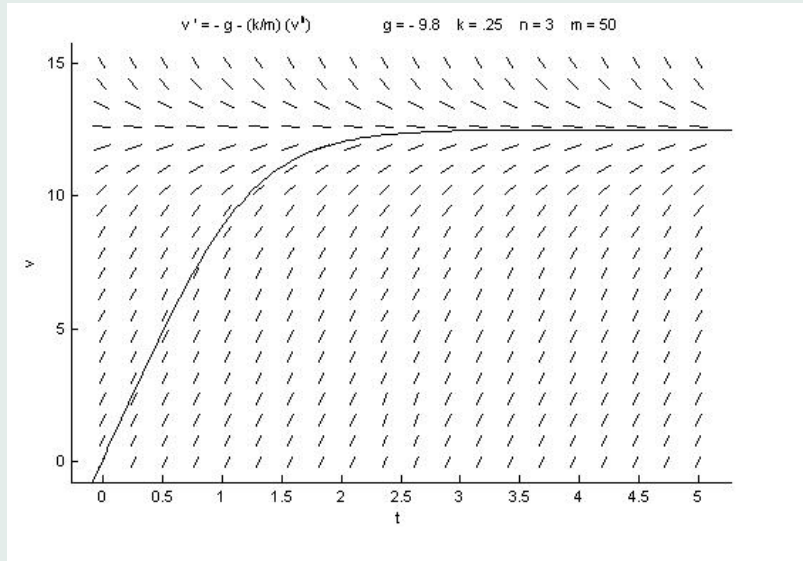
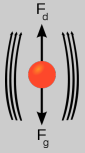


Figure 3:  $n = 3$ , Terminal Velocity Approx = 12.5

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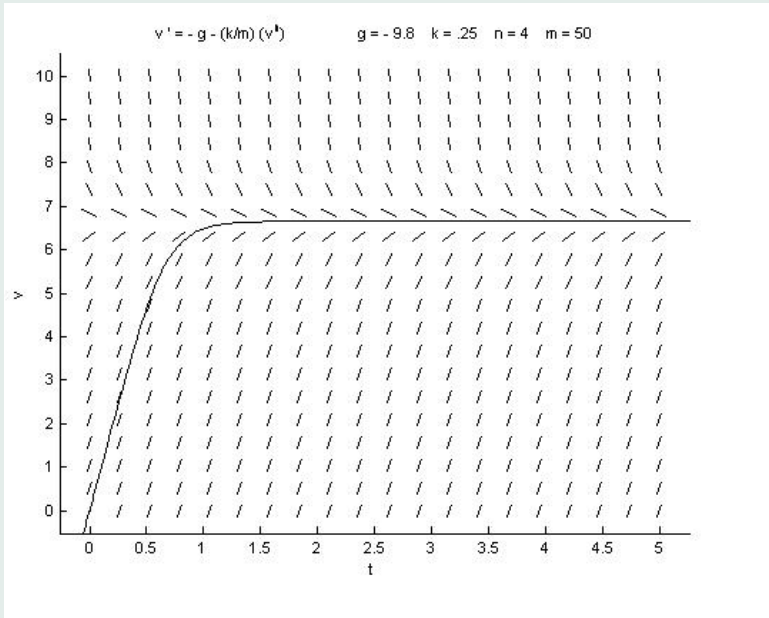
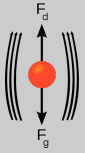


Figure 4:  $n = 4$ , Terminal Velocity Approx = 6.6

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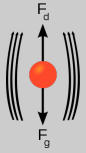
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the maximum terminal velocities of objects of differing reynolds numbers, and we should note the real-world importance in recognizing these differences for applications in aircrafts, ships, cars, buildings... and just about everything that we strive to build with aerodynamic and structural efficiency in mind.

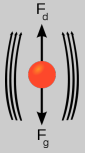
## 7. Conclusion

Our new model is indeed much improved over the old, and through differential equations we have managed to show the vast error associated with applying a linear velocity dependence to free-falling objects with large reynolds numbers. Though, while this model is indeed superior, it still does not in fact truly represent the sum of all physics and knowledge. Our model does not take into account various non-static factors that may be found within the variable  $k$ , and it assumes homogenous fluid density and viscosity and so forth.

In conclusion, it is truly important for us to understand the vast errors that may be made by blindly sticking with a simplifying assumption to a system, yet we should be thankful that we are still able to achieve accurate solutions for a fair amount of cases that the assumption is made upon.

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